Magnetic instability in a rotating layer at highly eccentric positions of the critical level

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Model

- A horizontal fluid layer between $[-d/2, d/2]$, rotating with the rate Ω_0, having density ρ, magnetic diffusivity η and permeability μ.
- Effects caused by the field $\mathbf{B_0} = \mathcal{B}_0 \tanh[\gamma(z - z_0)]\hat{\mathbf{y}}$ are of interest. The parameter γ enables to modify the field gradient and to localize it.
- The basic-state velocity $\mathbf{U_0} = 0$. In the magnetostrophic approximation, the linear stability problem is described

$$2\rho\mathbf{\Omega_0} \times \mathbf{u} = -\nabla p + \frac{1}{\mu}(\nabla \times \mathbf{B_0}) \times \mathbf{b} + \mathbf{j} \times \mathbf{B_0},$$

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B_0}) + \eta \nabla^2 \mathbf{b},$$

$$\nabla \cdot \mathbf{u} = 0, \qquad \nabla \cdot \mathbf{b} = 0,$$

- Stress-free, electrically perfectly conducting boundaries are considered.

Model

- Taking the z-component of the induction equation, the z-components of $\nabla\times$induction equation, $\nabla\times$Navier-Stokes equation, the z-component of $\nabla\times\nabla\times$ Navier-Stokes equation, a system of partial differential equations for $u_z,\ \omega_z,\ b_z,\ j_z$ is obtained.
- Non-dimensionalisation is performed taking: d as the length-scale, $\mathcal{B}_0$ as the magnetic field characteristic strength, $\tau_s = \dfrac{2\Omega_0 d^2 \mu\rho}{{\mathcal{B}_0}^2}$ as the time-scale.
- The solution is sought as

$$\begin{aligned}\{u_z,\ \omega_z,\ b_z,\ j_z\} &= \{u,\ \omega,\ b,\ j\}(z)\ exp[\mathrm{i}(k_x x + k_y y) + st] \\ &= \{u,\ \omega,\ b,\ j\}(z)\ exp[\mathrm{i}k\zeta + st],\end{aligned}$$

where $k_x = k\cos\phi = \beta k, \quad k_y = k\sin\phi = \alpha k$, and $s = \lambda + \mathrm{i}\varpi$.

Model

- The system of ordinary differential equations is to be solved $\left(\mathrm{D}=\frac{\mathrm{d}}{\mathrm{d}z}\right)$

$$\mathrm{D}u = -\beta \mathrm{i}k[\mathrm{D}B_0(z)]b - \alpha \mathrm{i}k[B_0(z)]j,$$
$$\mathrm{D}\omega = \alpha \mathrm{i}k[B_0(z)](\mathrm{D}^2 - k^2)b - \alpha \mathrm{i}k[\mathrm{D}^2 B_0(z)]b,$$
$$sb = \mathrm{i}k\alpha[B_0(z)]u + \frac{1}{\Lambda}(\mathrm{D}^2 - k^2)b,$$
$$sj = \mathrm{i}k\alpha[B_0(z)]\omega - \mathrm{i}k\beta[\mathrm{D}B_0(z)]u + \frac{1}{\Lambda}(\mathrm{D}^2 - k^2)j.$$

The function $B_0(z) = \tanh[\gamma(z - z_0)]$ features a zero point in $[-1/2, 1/2]$, by what the critical level condition $\mathbf{k} \cdot \mathbf{B_0} = 0$ is satisfied.

$$\mathrm{D}B_0(z) = \frac{\gamma}{\cosh^2[\gamma(z - z_0)]}, \quad \mathrm{D}^2 B_0(z) = -2\gamma[B_0(z)][\mathrm{D}B_0(z)],$$

$$\Lambda = \frac{\tau_\eta}{\tau_s} = \frac{\sigma \mathcal{B}_0{}^2}{2\Omega_0 \rho}.$$

Numerical results

- The numerics was performed for a rotating stratified layer.
  Density stratification was measured by Rayleigh number R.
- The layer was permeated by the field $\mathbf{B_0} = \tanh[\gamma(z - z_0)]\hat{\mathbf{y}}, \; \gamma = 80$.
  That means a strong field gradient localized to the thin shear region around the critical point $z = z_0$.
- Both, bulk and localized (predominantly magnetically driven) modes of convection were possible. Preference depended on the critical level position with respect to a perfectly conducting boundary.
- The critical layer evolved when the critical level was close enough at the (bottom, $z = -0.5$) perfectly conducting boundary, $z_0 \leq -0.388$.
- The stationary, critical-layer mode did not depend on the electromagnetic nature of the distant boundary.
  It was identified with the tearing mode.

Numerical results

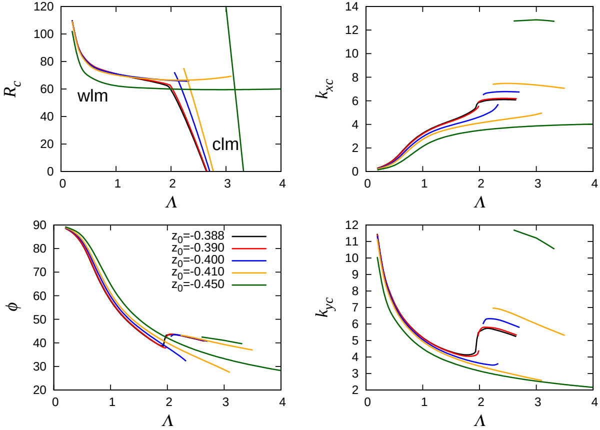

Figure: Dependences of critical parameters R_c, ϕ, k_{xc}, k_{yc} on Elsasser number Λ for modes in the layer permeated by the field $\mathbf{B_0} = \mathcal{B}_0 \tanh[\gamma(z - z_0)]\hat{\mathbf{y}}$, $\gamma = 80$, and enclosed by perfectly conducting boundaries.

Numerical results

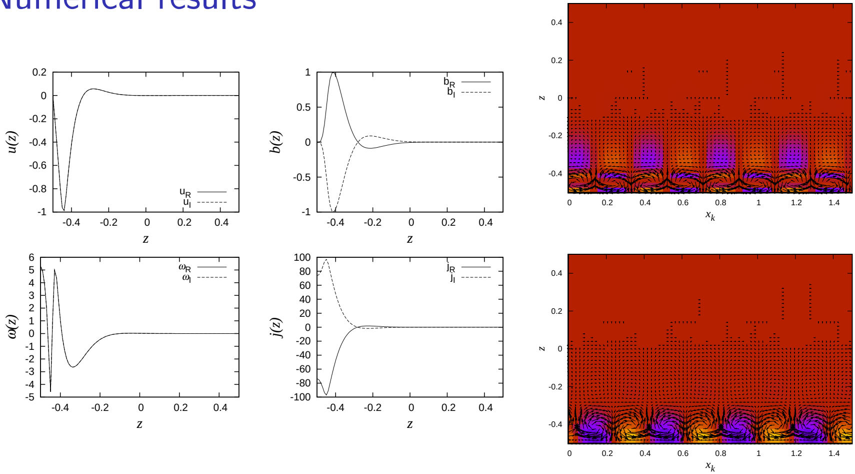

Figure: Magnetically driven critical-layer mode at $\gamma = 80$, $z_0 = -0.45$ and mixed boundaries.

Analytical approach, $\gamma >> 1$

- $$\begin{pmatrix} u(z) \\ \omega(z) \\ b(z) \\ j(z) \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & k^2 & 0 & 0 \\ 0 & 0 & \mathrm{i}k & 0 \\ 0 & 0 & 0 & \dfrac{\mathrm{i}}{k} \end{pmatrix} \begin{pmatrix} u(z) \\ \omega(z) \\ b(z) \\ j(z) \end{pmatrix}.$$

- $$\begin{aligned} \mathrm{D}u &= \beta k^2[\mathrm{D}B_0(z)]b + \alpha[B_0(z)]j, \\ \mathrm{D}\omega &= -\alpha[B_0(z)](\mathrm{D}^2 - k^2)b + \alpha[\mathrm{D}^2 B_0(z)]b, \\ sb &= \alpha[B_0(z)]u + \frac{1}{\Lambda}(\mathrm{D}^2 - k^2)b, \\ sj &= \alpha^2 k^4[B_0(z)]\omega - \beta k^2[\mathrm{D}B_0(z)]u + \frac{1}{\Lambda}(\mathrm{D}^2 - k^2)j. \end{aligned}$$

Solution in the outer region, $\Lambda \to \infty$

- Spatially changeless field, no driving mechanism for motions: $\mathrm{D}u \approx 0$ and $\mathrm{D}\omega \approx 0$, $s = 0$.
- $$\begin{aligned} i. \quad 0 &\approx \frac{\beta k^2 \gamma \ b}{\cosh^2[\gamma(z - z_0)]} + \alpha \tanh[\gamma(z - z_0)] j, \\ ii. \quad 0 &\approx \alpha \tanh[\gamma(z - z_0)] \left(\mathrm{D}^2 - k^2 + \frac{2\gamma^2}{\cosh^2[\gamma(z - z_0)]} \right) b, \\ iii. \quad 0 &\approx \alpha \tanh[\gamma(z - z_0)] u, \\ iv. \quad 0 &\approx \alpha^2 k^4 \tanh[\gamma(z - z_0)] \omega - \frac{\beta k^2 \gamma \ u}{\cosh^2[\gamma(z - z_0)]} \end{aligned}$$
- Boundary conditions:

$$u\left(\frac{1}{2}\right) = 0, \quad b\left(\frac{1}{2}\right) = 0, \quad \mathrm{D}j\left(\frac{1}{2}\right) = 0.$$

Solution in the outer region

- Since ($z \neq z_0$), from (*ii.*) we have
$$\mathrm{D}^2 b = \left(k^2 - \frac{2\gamma^2}{\cosh^2[\gamma(z - z_0)]} \right).$$
- By the change of the independent variable $w = \tanh[\gamma(z - z_0)]$ we get the associated Legendre differential equation
$$\frac{\mathrm{d}}{\mathrm{d}w}\left[(1 - w^2)\frac{\mathrm{d}b}{\mathrm{d}w}\right] + \left[2 - \frac{k^2}{\gamma^2(1 - w^2)}\right] b = 0$$
- Taking the presumptions and boundary conditions into account, it is found:
$$b(z) = kP_1^{-\frac{k}{\gamma}}(\tanh[\gamma(z - z_0)]),$$
$$j(z) = -\frac{\beta k \gamma P_1^{-\frac{k}{\gamma}}(\tanh[\gamma(z - z_0)])}{\alpha \cosh[\gamma(z - z_0)] \sinh[\gamma(z - z_0)]},$$
$$u(z) = 0, \qquad \omega(z) = 0.$$

Solution in the inner region (critical layer)

- Rescaling: $\chi = \gamma(z - z_0)$ and $\frac{\mathrm{d}}{\mathrm{d}z} = \gamma \frac{\mathrm{d}}{\mathrm{d}\chi} = \gamma \mathrm{D}$.
- A longwavelength solution relative to the width of the current layer is expected: $\kappa = \frac{k}{\gamma}$ and $\gamma \gg k$ (> 0).
- Performing the substitutions, we obtain

$$
\begin{aligned}
i.\quad \gamma \mathrm{D} u &= \beta \frac{k^2}{\gamma^2} \gamma \mathrm{D} B_0(\chi) b + \alpha B_0(\chi) j, \\
ii.\quad \gamma \mathrm{D} \omega &= -\alpha B_0(\chi) \left(\gamma^2 \mathrm{D}^2 - \frac{k^2}{\gamma^2} \right) b + \alpha \gamma^2 \mathrm{D}^2 B_0(\chi) b, \\
iii.\quad sb &= \alpha B_0(\chi) u + \frac{1}{\Lambda} \left(\gamma^2 \mathrm{D}^2 - \frac{k^2}{\gamma^2} \right) b, \\
iv.\quad sj &= \alpha^2 \frac{k^4}{\gamma^4} B_0(\chi) \omega - \beta \frac{k^2}{\gamma^2} \gamma \mathrm{D} B_0(\chi) u + \frac{1}{\Lambda} \left(\gamma^2 \mathrm{D}^2 - \frac{k^2}{\gamma^2} \right) j.
\end{aligned}
$$

Solution in the inner region

- The equations are to be solved for the marginal stability state, $s = 0$.
- Boundary conditions: $u(\chi_B) = 0$, $b(\chi_B) = 0$, $\mathrm{D}j(\chi_B) = 0$, where $\chi_B = \chi\left(-\frac{1}{2}\right) = -\gamma\left(\frac{1}{2} + z_0\right)$.
- The following expansions are convenient to obtain a balance in the equations:

$$u = u_0 + \frac{u_1}{\gamma} + \frac{u_2}{\gamma^2} + \cdots,$$

$$\omega = \gamma\omega_0 + \omega_1 + \frac{\omega_2}{\gamma} + \cdots,$$

$$b = b_0 + \frac{b_1}{\gamma} + \frac{b_2}{\gamma^2} + \cdots,$$

$$j = \gamma j_0 + j_1 + \frac{j_2}{\gamma} + \cdots,$$

$$\Lambda = \gamma^2\Lambda_0.$$

Solution in the inner region

- For the primary balance it is obtained

$$\begin{aligned}
&i. \quad \mathrm{D}u_0 = \alpha B_0(\chi) j_0,\\
&ii. \quad \mathrm{D}\omega_0 = -\alpha B_0(\chi)\mathrm{D}^2 b_0 + \alpha \mathrm{D}^2 B_0(\chi) b_0,\\
&iii. \quad \mathrm{D}^2 b_0 = -\alpha \Lambda_0 B_0(\chi) u_0,\\
&iv. \quad \mathrm{D}^2 j_0 = 0,
\end{aligned}$$

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$$\begin{aligned}
&iv. \ \rightarrow \ j_0 = C_j, \quad C_j \in \mathbb{R},\\
&i. \ \rightarrow \ u_0 = C_j \alpha \left[\ln\left(\cosh(\chi)\right) - \ln\left(\cosh(\chi_B)\right)\right],\\
&iii. \ \rightarrow\\
&\mathrm{D}b_0 = -C_j \alpha^2 \Lambda_0 \left[\frac{1}{2}\ln^2\left(\cosh(\chi)\right) - \ln\left(\cosh(\chi)\right)\ln\left(\cosh(\chi_B)\right)\right] + C_b,\\
&C_b \in \mathbb{R}.
\end{aligned}$$

Solution in the inner region

- Demand on marginal instability to be structurally simplest possible to determine C_b.
- Setting $\mathrm{D}b_0 = 0$:

$$0 = \ln^2(\cosh(\chi)) - 2\ln(\cosh(\chi))\ln(\cosh(\chi_B)) - \frac{2C_b}{C_j\alpha^2\Lambda_0}.$$

- Minimum magnetic energy for instability

$$\Lambda_{0min} = -\frac{2C_b}{C_j\alpha^2\ln^2(\cosh(\chi_B))}.$$

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$$\mathrm{D}b_0 = -C_j\alpha^2\Lambda_0 \left[\frac{1}{2}\ln^2(\cosh(\chi)) - \ln(\cosh(\chi))\ln(\cosh(\chi_B)) + \frac{1}{2}\ln^2(\cosh(\chi_B))\right].$$

Solution in the inner region

- $$\begin{aligned} \mathrm{D}b_0 \approx -C_j\alpha^2\Lambda_0 \quad & [a_0 + a_1(\chi-\chi_0) + a_2(\chi-\chi_0)(\chi-\chi_1) + \\ & a_3(\chi-\chi_0)(\chi-\chi_1)(\chi-\chi_2) \\ & +a_4(\chi-\chi_0)(\chi-\chi_1)(\chi-\chi_2)(\chi-\chi_3) + C_A] \\ \approx -C_j\alpha^2\Lambda_0 \quad & \left[A_0 + C_A + A_1\chi + A_2\chi^2 + A_3\chi^3 + A_4\chi^4\right]. \end{aligned}$$
- $\chi_0 = \chi_B - \frac{h}{2}$, $\chi_1 = \chi_0 + h$, $\chi_2 = -1$, $\chi_3 = 1$, $\chi_4 = \chi_0 + 4h$, $h = -\frac{\chi_B}{2}$. The constant C_A is chosen to obtain $P_4(\pm\chi_B) = 0$.
- $$\begin{aligned} \mathrm{D}\omega_0 & \approx -\chi \mathrm{D}^2 b_0 \\ & \approx C_j\alpha^2\Lambda_0\left[A_1\chi + 2A_2\chi^2 + 3A_3\chi^3 + 4A_4\chi^4\right]. \end{aligned}$$

Solution in the inner region

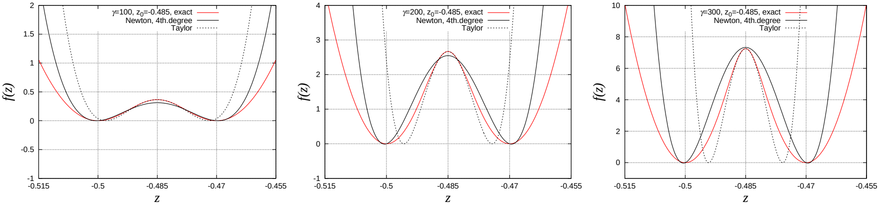

Figure: The function $0.5\ln^2\left(\cosh[\gamma(z+0.485)]\right) - \ln\left(\cosh[\gamma(z+0.485)]\right)\ln\left(\cosh[\gamma(-0.015)]\right) + 0.5\ln^2\left(\cosh[\gamma(-0.015)]\right)$ and its approximations by the Newton polynomial of the fourth degree and by the first terms of Taylor expansions around $z_0 = -0.485$ of both parts of the original function for $\gamma = 100,\ 200,\ 300$.

Solution in the inner region (critical layer)

- In the highest order, the solutions are:

$$
\begin{aligned}
u(z) &\approx C_j\alpha\left[\ln\left(\cosh\gamma(z-z_0)\right)-\ln\left(\cosh[\gamma(-0.5-z_0)]\right)\right]\\
\omega(z) &\approx \frac{C_j {k_c}^2\alpha^3\Lambda_c}{\gamma^2}\left[A_1\gamma(z-z_0)+2A_2[\gamma(z-z_0)]^2\right.\\
&\left.+3A_3[\gamma(z-z_0)]^3+4A_4[\gamma(z-z_0)]^4\right]+C,\\
b(z) &\approx -\frac{C_j k_c\alpha^2\Lambda_c}{\gamma^2}\left[(A_0+C_A)\gamma(z-z_0)+\frac{A_1}{2}[\gamma(z-z_0)]^2\right.\\
&\left.+\frac{A_2}{3}[\gamma(z-z_0)]^3+\frac{A_3}{4}[\gamma(z-z_0)]^4+\frac{A_4}{5}[\gamma(z-z_0)]^5+C_B\right],\\
j(z) &\approx \frac{C_j}{k_c}\gamma, \qquad C,\ C_j,\ C_A,\ C_B\ \in\mathbb{R}.
\end{aligned}
$$

Result

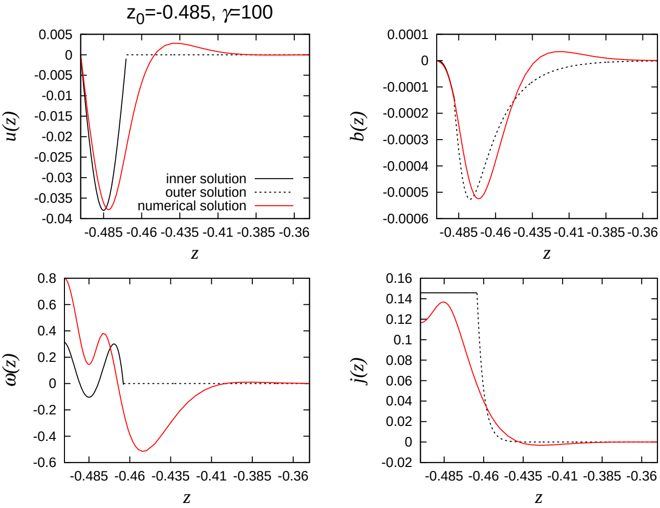

Figure: Analytically and numerically obtained solutions for the critical-layer mode (tearing mode) for $z_0 = -0.485$. $k_c = 50.606$, $\phi = 37.026$, $\Lambda_c = 6.585$.

Conclusion

- Main features of the analytically obtained solutions are in a good qualitative accordance with the numerical ones.

The appropriateness of the simplifying physical assumptions made in each region was confirmed.