Numerical modeling of dynamic rupture propagation

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introduction

FEM (Finite-Element Method)

TSN (Traction at Split Nodes) approach
and its implementation in FEM

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ADER-DGM
(Arbitrarily high-order DERivative
Discontinuous Galerkin Method)

implementation of
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3D numerical comparison **FEM-ASA vs ADER-DGM**

effect of the numerical nucleation zone on the rupture propagation

FEM (Finite-Element Method)

displacement formulation of equation of motion - D-EqM

$$\rho \ddot{u}_i = \sigma_{ij},_j \qquad \left(\sigma_{ij} = \lambda \delta_{ij} u_k,_k + \mu (u_i,_j + u_j,_i) \right)$$

displacement formulation of equation of motion - D-EqM

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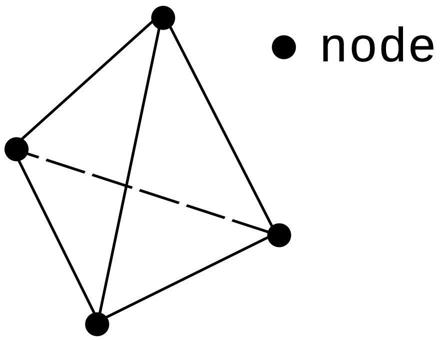
choose an element with n nodes

e.g. tetrahedron with 4 nodes



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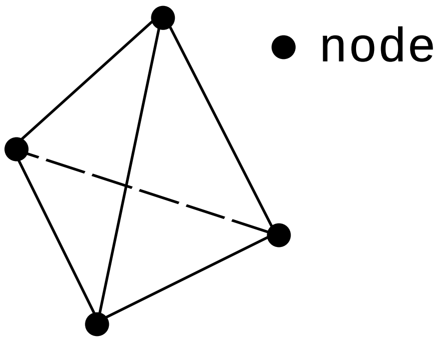

choose shape functions and approximation to displacement in the element

$$u_i\left(x_j, t\right) = s^k\left(x_j\right) U_i^k(t) \quad ; \quad k = 1, \ldots, n$$

unique displacements U_i^k at nodes

continuity of displacement u_i at a contact of elements

multiply D-EqM by the shape functions

$$\rho \ddot{u}_i s^k = \sigma_{ij,j} s^k \quad ; \quad k = 1, \ldots, n$$

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integrate over an element

$$\int_{\Omega^e} \rho \ddot{u}_i s^k dV = \int_{\Omega^e} \sigma_{ij,j} s^k dV$$

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integrate the r.h.s. by parts

$$\int_{\Omega^e} \rho \ddot{u}_i s^k dV = \int_{\Omega^e} \sigma_{ij} s,_j^k dV + \int_{\partial \Omega^e} T_i s^k dS$$

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$$\rho \ddot{u}_i \, s^k = \sigma_{ij,j} \, s^k \quad ; \quad k = 1, \ldots, n$$

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$$\mathbf{M}^{\mathrm{e}} \ddot{\mathbf{u}}^{\mathrm{e}} = \mathbf{r}^{\mathrm{e}} + \text{surface-traction boundary term}$$

$\mathbf{M}^{\mathrm{e}}$: local mass matrix

$\ddot{\mathbf{u}}^{\mathrm{e}}$: local vector of nodal displacements

$\mathbf{r}^{\mathrm{e}}$: local vector of restoring forces

$$\mathbf{M}^e \ddot{\mathbf{u}}^e = \mathbf{r}^e + \text{surface-traction boundary term}$$

assemble all elements covering volume Ω closed by surface $\partial\Omega$

$$\partial\Omega = \partial\Omega_{FS} \cup \partial\Omega_{\text{Dirichlet}}$$

$$\mathbf{M}\,\ddot{\mathbf{u}} = \mathbf{r} + 0$$

- $\mathbf{M}$: global mass matrix
- $\ddot{\mathbf{u}}$: global vector of nodal displacements
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$$\mathbf{M}^e \ddot{\mathbf{u}}^e = \mathbf{r}^e + \text{surface-traction boundary term} \sim \int_{\partial\Omega^e} T_i \, s^k \, dS$$

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the theoretical boundary term vanishes
- at a contact of two elements due to traction continuity
- at the free surface due to zero traction

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the theoretical boundary term vanishes
- at a contact of two elements due to traction continuity
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in fact, however, the final discretization does not give
- the traction continuity at a contact of two elements
- zero traction at the free surface

and thus the zero boundary term in the global equation

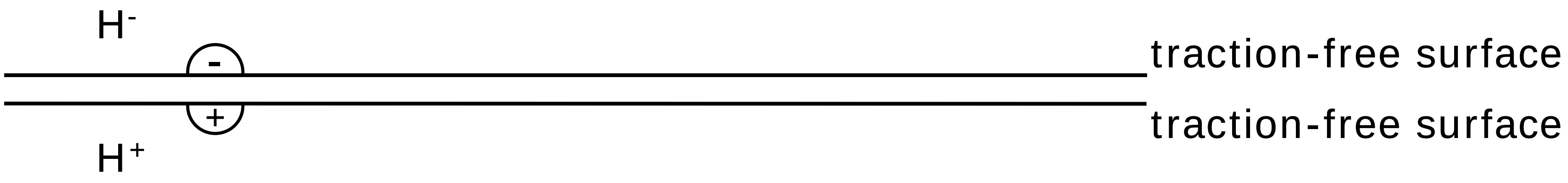
they are just low-order approximated

TSN (Traction at Split Nodes) approach

and

its implementation in FEM

elastic halfspaces H^- and H^+ do not interact



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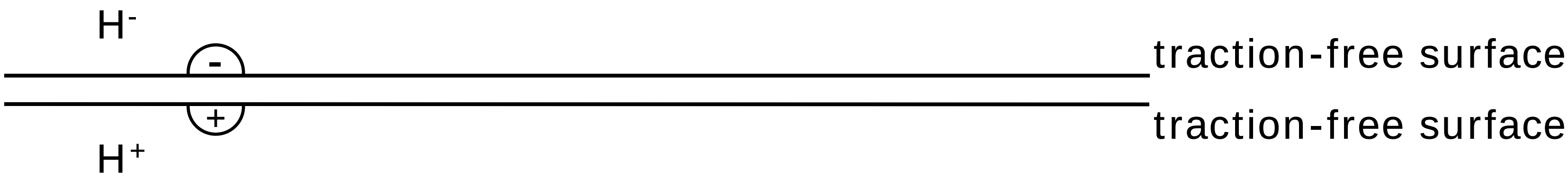
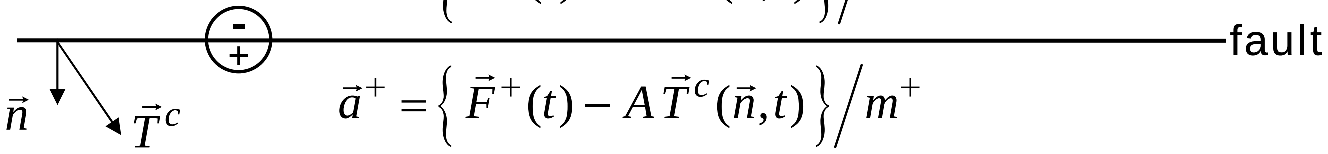


halfspaces coupled by a constraint surface traction

$$\vec{a}^{\,-} = \left\{ \vec{F}^{-}(t) + A\,\vec{T}^{c}(\vec{n},t) \right\} \Big/ m^{-}$$



$$\vec{a}^{\,+} = \left\{ \vec{F}^{+}(t) - A\,\vec{T}^{c}(\vec{n},t) \right\} \Big/ m^{+}$$

enforcement of the frictional boundary condition on the fault

if $\left|\vec{T}_{sh}^{\,ct}(t)\right| \leq S(t)$ then $\vec{T}^{\,c}(t) = \vec{T}^{\,ct}(t)$

no slip

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if $\left|\vec{T}_{sh}^{\,ct}(t)\right| \leq S(t)$ then $\vec{T}^{\,c}(t) = \vec{T}^{\,ct}(t)$

no slip

if $\left|\vec{T}_{sh}^{\,ct}(t)\right| > S(t)$ then $\vec{T}_{sh}^{\,c}(t) = \vec{T}_{sh}^{\,f}(t)$

$$\vec{T}_{n}^{\,c}(t) = \vec{T}_{n}^{\,ct}(t)$$

$$D\vec{v}\,(t + dt/2) = A\frac{m^{-} + m^{+}}{m^{-}m^{+}}\left[\,\vec{T}^{\,ct} - \vec{T}^{\,f}\,\right]dt$$

we apply our

adaptive smoothing algorithm (ASA)

to the trial traction

before it is used for updating the slip rate

in order

to reduce spurious high-frequency oscillations

of the slip rate

ADER-DGM
(Arbitrarily high-order DERivative Discontinuous Galerkin Method)

velocity-stress formulation of equation of motion - VS-EqM

$$\dot{\sigma}_{ij} - \lambda v_{k,k}\delta_{ij} - \mu(v_{i,j} - v_{j,i}) = 0$$

$$\rho\dot{v}_i - \sigma_{ij,j} = 0$$

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define a vector of unknown variables

$$Q = \left(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \sigma_{yz}, \sigma_{zx}, v_x, v_y, v_z \right)^T$$

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VS-EqM in the matrix form

$$\dot{Q}_p + A_{pq} Q_{q,x} + B_{pq} Q_{q,y} + C_{pq} Q_{q,z} = 0$$

A, B, C space-dependent matrices include material properties

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A, B, C space-dependent matrices include material properties

tetrahedral element (e.g.)

$$(Q_h)_p = \hat{Q}_{pk}(t)\ \Phi_k(x_j)$$

└── polynomial basis functions of an optional degree

multiply VS-EqM by a test function and integrate over an element volume

$$\int_{\Omega^e} \dot{Q}_p \, \Phi_k \, dV \; - \; \int_{\Omega^e} \Big(A_{pq} \, Q_{q,x} \; + \; B_{pq} \, Q_{q,y} \; + \; C_{pq} \, Q_{q,z} \Big) \Phi_k \, dV \; = \; 0$$

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integrate the 2nd integral by parts

$$\int_{\Omega^e} \dot{Q}_p \, \Phi_k \, dV \; - \; \int_{\Omega^e} \left(\Phi_{k,x} \, A_{pq} \; + \; \Phi_{k,y} \, B_{pq} \; + \; \Phi_{k,z} \, C_{pq} \right) Q_q \, dV \; + \; \int_{\partial\Omega^e} \Phi_k \, F_p \, dS \; = \; 0$$

numerical flux introduced
because Q_h may be discontinuous
at an element boundary

multiply VS-EqM by a test function and integrate over an element volume

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numerical flux introduced
because Q_h may be discontinuous
at an element boundary

Riemann problem – an evolution physically continuous problem
with initial discontinuous approximation of unknowns
across an interface

to find a flux such that
continuity of particle velocity and traction
at an element boundary is assured

implementation of
the frictional boundary conditions
in ADER-DGM

imposing frictional boundary condition at S (the 2D example)

- frictional traction

$$\tilde{\sigma}_{xy} = \sigma_{xy}^{G} + \delta\sigma_{xy}$$

imposed at S

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- consequently

$$\tilde{Q}\left(S^{\pm}\right) = Q^{G} + \left(0,\ 0,\ \delta\sigma_{xy},\ 0,\ \pm\frac{c_s}{\mu}\delta\sigma_{xy} \right)^{T}$$

and thus

$$\tilde{v}_{y}^{\pm} = v_{y}^{G} \pm \frac{c_s}{\mu}\left(\tilde{\sigma}_{xy} - \sigma_{xy}^{G}\right)$$

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- the slip rate is

$$\Delta\tilde{v}_{y} = \frac{2c_s}{\mu}\left(\tilde{\sigma}_{xy} - \sigma_{xy}^{G}\right)$$

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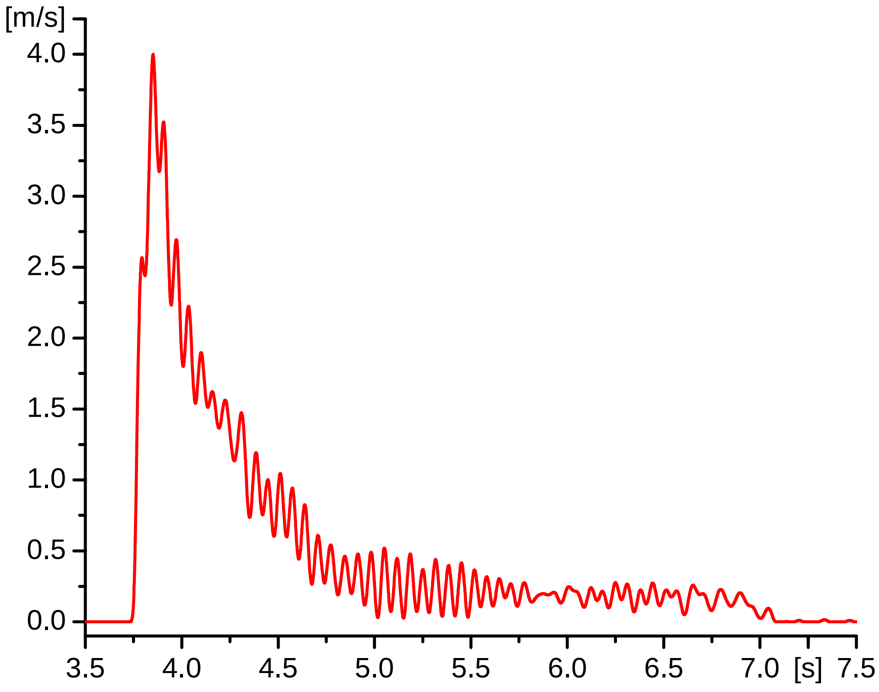
- the slip rate is

$$\Delta\tilde{v}_y = \frac{2c_s}{\mu}\left(\tilde{\sigma}_{xy} - \sigma_{xy}^{G}\right)$$

- all this means that the imposed frictional traction $\tilde{\sigma}_{xy}$, different from σ_{xy}^{G} , instantly and locally generates an imposed slip rate parallel to the fault

the problem of the spurious high-frequency oscillations

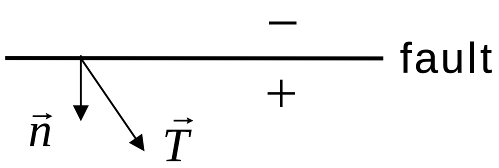


the problem of the spurious high-frequency oscillations

FEM

$$\vec{T}^{f}_{sh} = \vec{T}^{ct} - \frac{m^{-}m^{+}}{A\left(m^{-}+m^{+}\right)} D\vec{a}_{sh}$$

\- collinearity

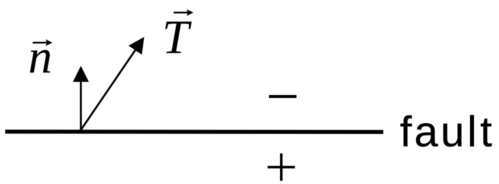


$$D\vec{v} = \vec{v}^{+} - \vec{v}^{-}$$

ADER-DG

$$\tilde{\sigma}_{xy} = \sigma^{G}_{xy} + \frac{\mu}{2c_s}\Delta\tilde{v}_y$$

\+ antiparallelism



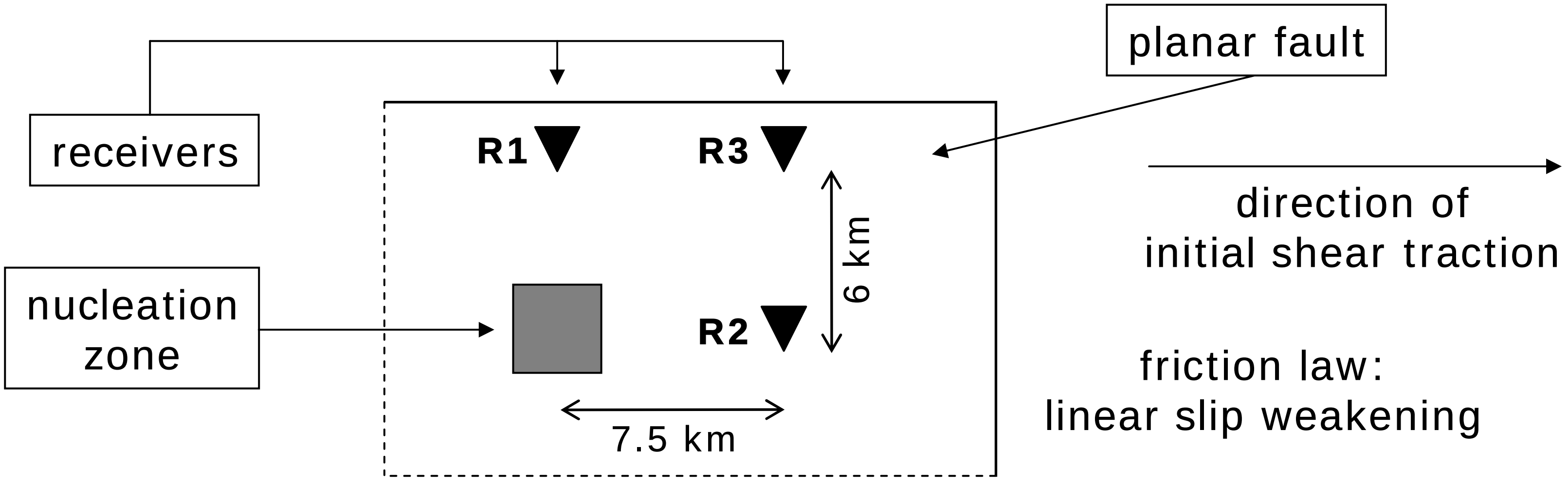
$$D\vec{v} = \vec{v}^{+} - \vec{v}^{-}$$

analogous

to the analytical formulation in BIEM

$\frac{\mu}{2c_s}\Delta\tilde{v}_y$ - instantaneous (high-frequency) stress response to the slip rate

3D numerical comparison **FEM-ASA** vs **ADER-DGM**



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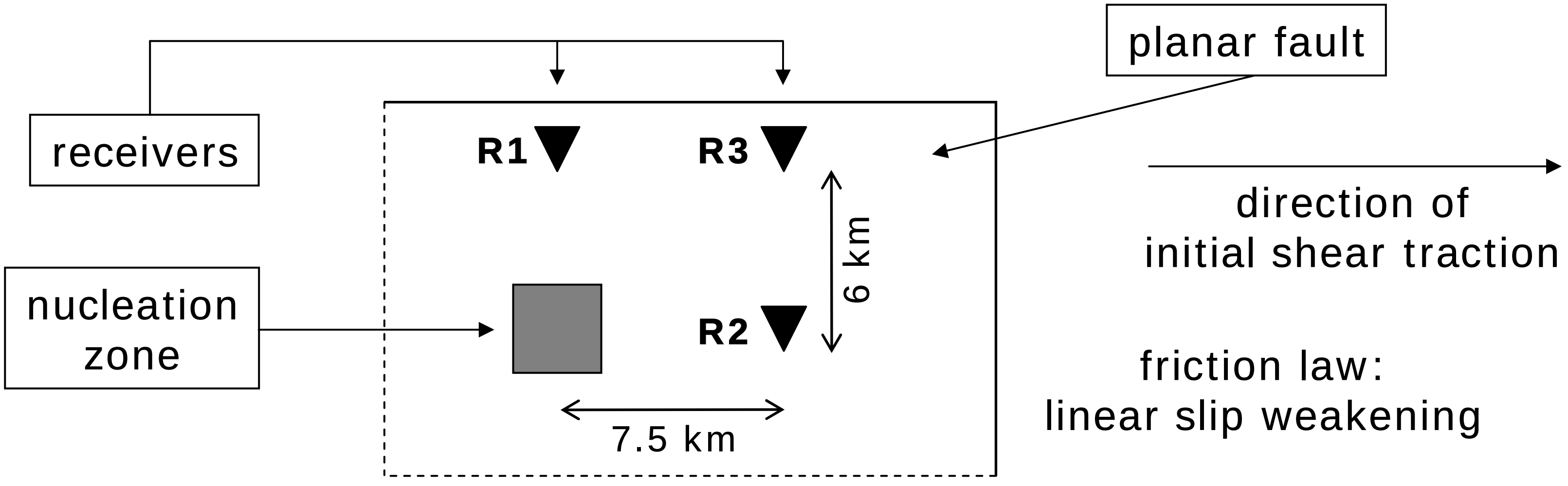
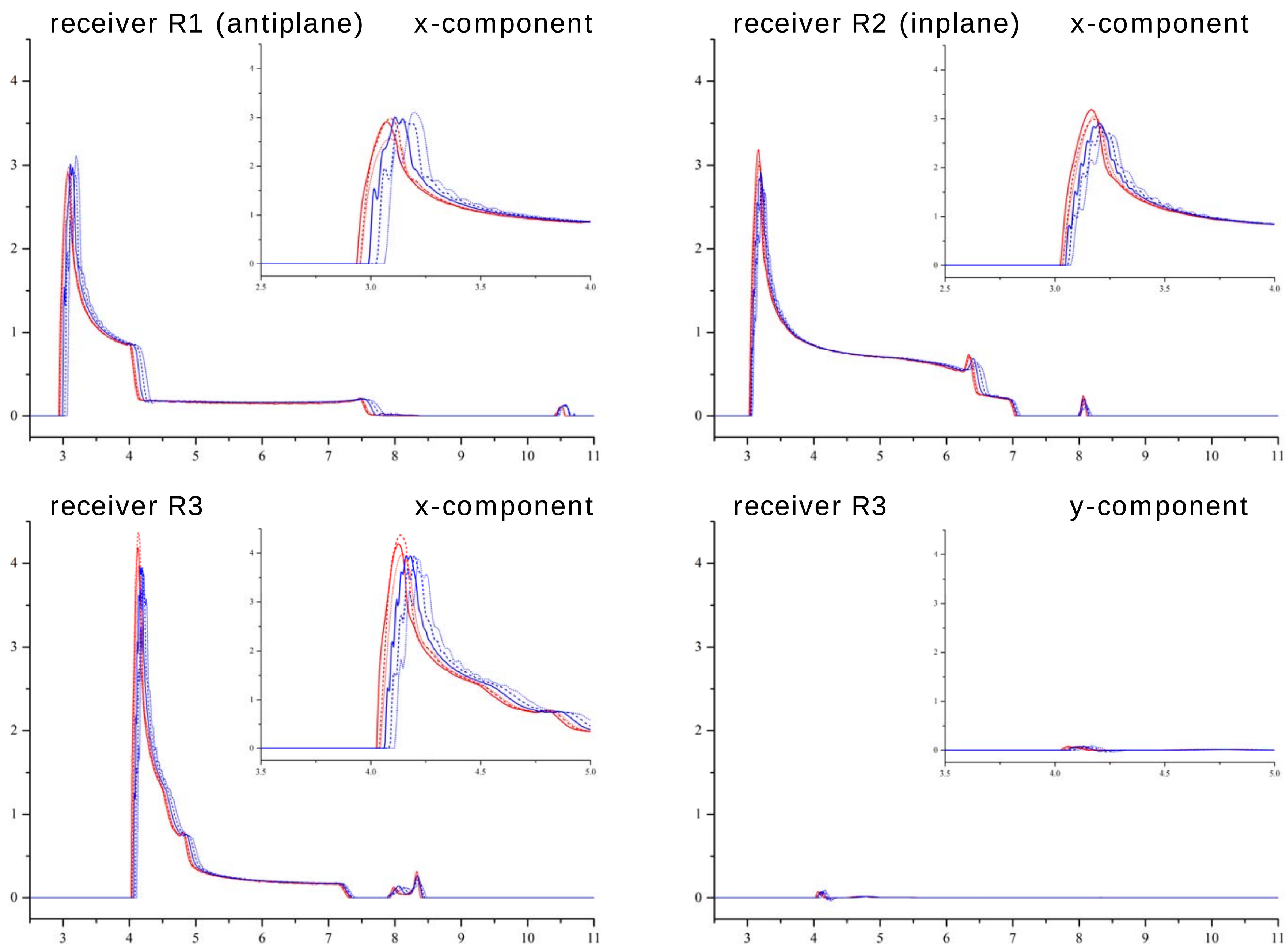


configuration		**sub-Rayleigh**	**supershear**
initial traction	τ_0	70.00 MPa	70.0 MPa
static friction	τ_s	81.33 MPa	73.5 MPa
dynamic friction	τ_d	63.00 MPa	63.0 MPa
critical distance		0.4 m	0.275 m
strength par.	$S = (\tau_s - \tau_0)/(\tau_0 - \tau_d)$	1.62	0.5
square nucleation zone, side length		3 km	3 km

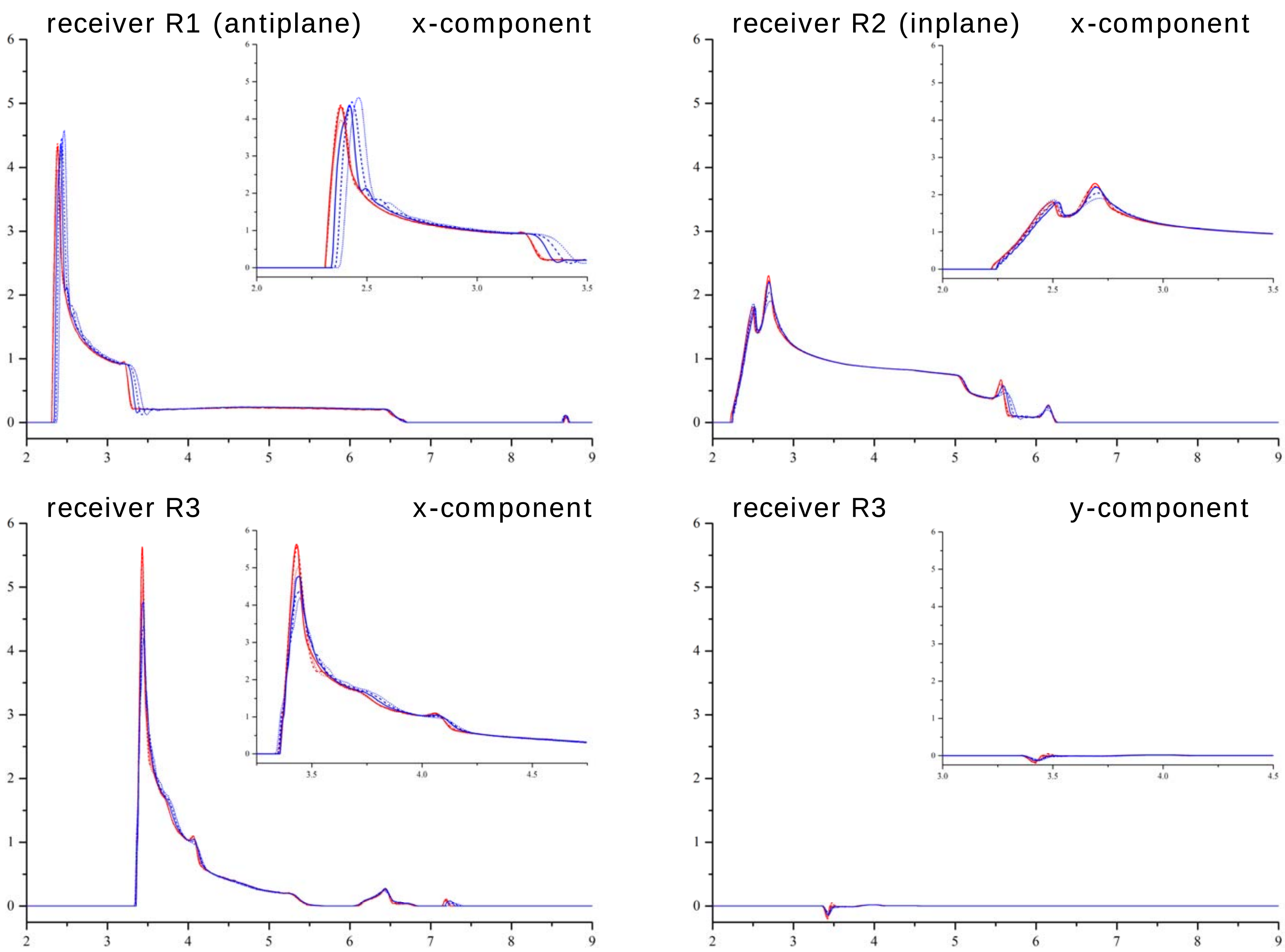
3D numerical comparison **FEM-ASA** vs **ADER-DGM**

		order	# of integ. points	element size	element type
red dotted	ADER-DG	O4	25	400 m	TET
red dashed	ADER-DG	O4	25	300 m	TET
red solid	ADER-DG	O4	25	200 m	TET
blue dotted	FEM-ASA	O2	4	100 m	HEX
blue dashed	FEM-ASA	O2	4	75 m	HEX
blue solid	FEM-ASA	O2	4	50 m	HEX

sub-Rayleigh rupture - slip-rate



supershear rupture - slip-rate



effect of the numerical nucleation zone

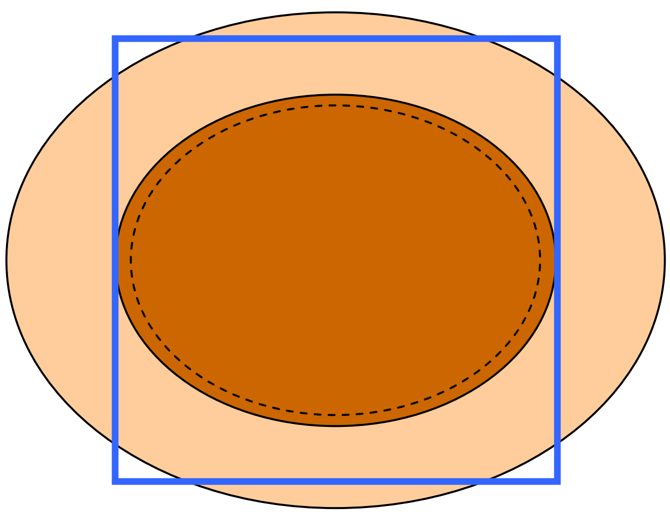
on the rupture propagation

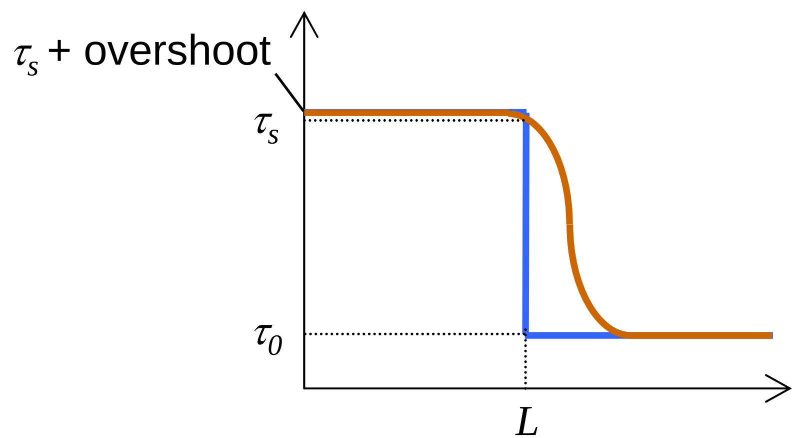
simulations by FEM-ASA

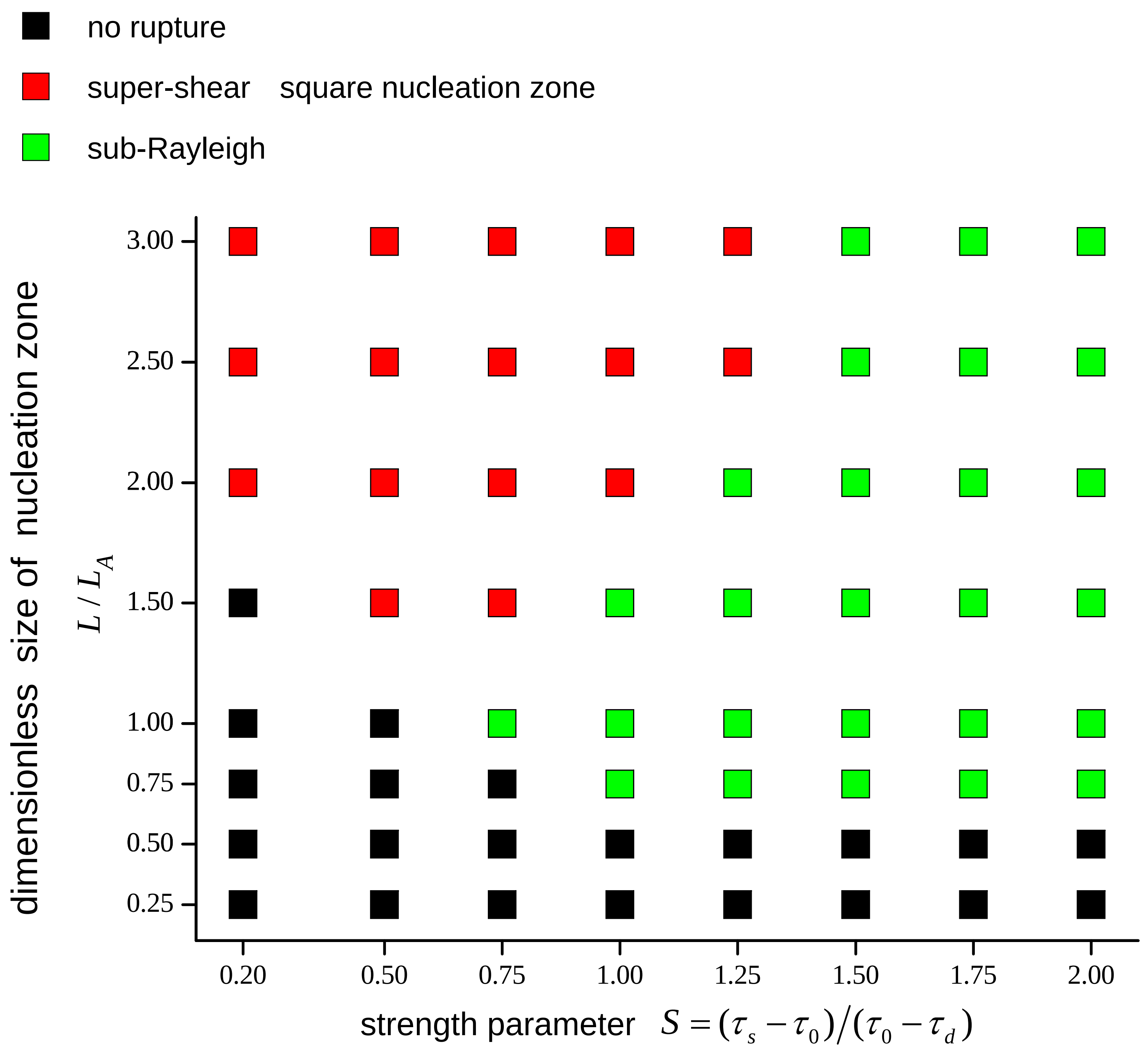
nucleation zones compared

square with an abrupt change in traction – for example SCEC

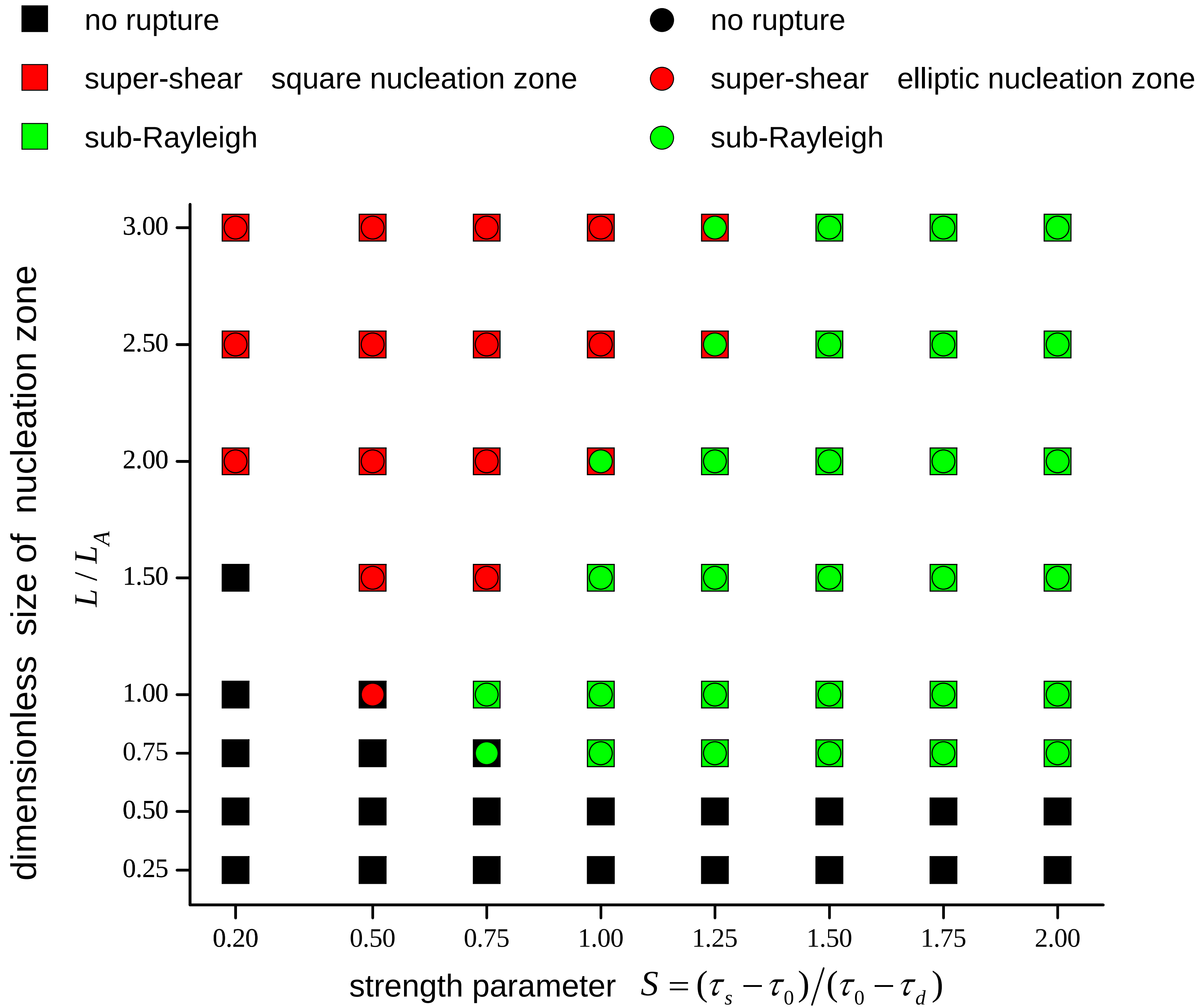
ellipse with a smooth change in traction







no rupture
super-shear square nucleation zone
sub-Rayleigh
dimensionless size of nucleation zone
L / L_A
3.00
2.50
2.00
1.50
1.00
0.75
0.50
0.25
0.20
0.50
0.75
1.00
1.25
1.50
1.75
2.00
strength parameter $S = (\tau_s - \tau_0)/(\tau_0 - \tau_d)$



no rupture
super-shear
sub-Rayleigh
square nucleation zone
no rupture
super-shear
sub-Rayleigh
elliptic nucleation zone
dimensionless size of nucleation zone L / L_A
3.00
2.50
2.00
1.50
1.00
0.75
0.50
0.25
0.20
0.50
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1.00
1.25
1.50
1.75
2.00
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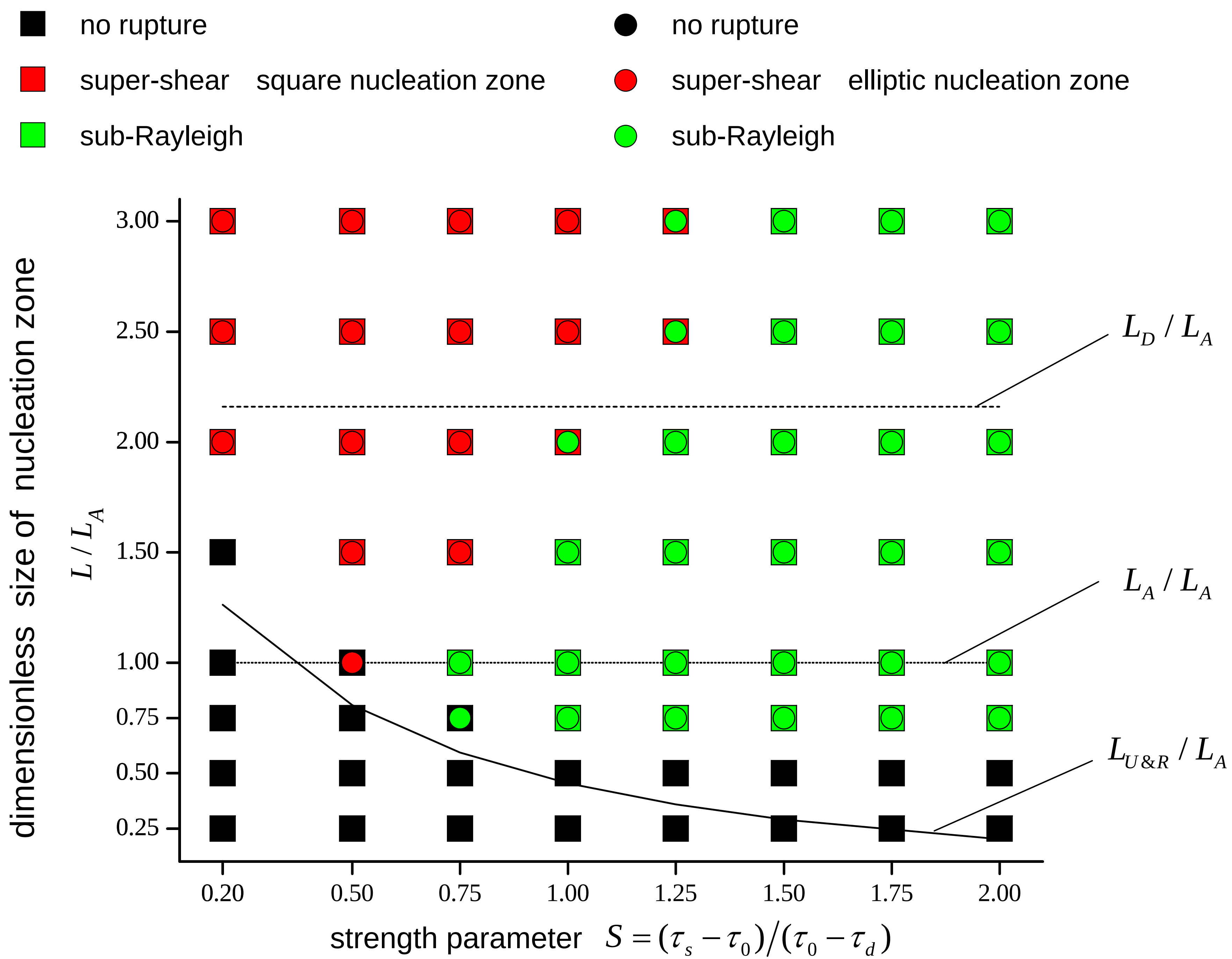

no rupture
super-shear
square nucleation zone
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no rupture
super-shear
elliptic nucleation zone
sub-Rayleigh
dimensionless size of nucleation zone
L / L_A
3.00
2.50
2.00
1.50
1.00
0.75
0.50
0.25
L_D / L_A
L_A / L_A
$L_{U\&R} / L_A$
0.20
0.50
0.75
1.00
1.25
1.50
1.75
2.00
strength parameter $S = (\tau_s - \tau_0)/(\tau_0 - \tau_d)$

thank you for your attention