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Basics of Continuum Mechanics

(unpublished lecture notes for students of geophysics)

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Preface

These notes only reproduce the first chapter of *Introduction to Theoretical Seismology*. This file was created for convenience of those who are not interested in seismology and only want to learn basic concepts of continuum mechanics. As in the preface to *Introduction to Theoretical Seismology* I want to stress that the notes are just transcription of what I originally hand-wrote on transparencies for students of the course *Theory of Seismic Waves* at Universität Wien in 2001. In other words, the material was not and is not intended as a standard introductory text on the topic.

The material is mainly based on the book by Aki and Richards (1980, 2002).

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1. BASIC RELATIONS OF CONTINUUM MECHANICS

1.1 Introduction

An application of a force to real object causes some deformation of the object, i.e., change of its shape. If the deformation is negligibly small, we can work with a concept of a rigid body. The rigid body retains a fixed shape under all conditions of applied forces. If the deformations are not negligible, we have to consider the ability of an object to undergo the deformation, i.e., its elasticity, viscosity or plasticity.

Here, we will restrict ourselves to the elastic behavior. For the purpose of the macroscopic description both the rigid and elastic bodies can be defined as a system of material particles (not atoms or molecules!). At the same time we assume a continuous distribution of mass – a continuum. In a continuum we assign values of material parameters to geometric points. Therefore, we can make use of the theory of continuous functions.

A value of a material parameter assigned to a geometric point represents an average value for such a volume of the material in which the real discontinuous (atomic or molecular) structure need not be considered.

In a rigid body, relative coordinates connecting all of the constituent particles remain constant, i.e., the particles do not undergo any relative displacements.

In an elastic body, the particles can undergo relative displacements if forces are applied. Concept of 'continuum' usually is used for description of elastic bodies and fluids. The elastic behavior or objects is a subject of the continuum mechanics.
1.2 Body forces

Non-contact forces proportional to mass contained in a considered volume of a continuum.
- forces between particles that are not adjacent; e.g., mutual gravitational forces
- forces due to the application of physical processes external to the considered volume; e.g., forces acting on buried particles of iron when a magnet is moving outside the considered volume

Let \( \vec{f}(\vec{x}, t) \) be a body force acting per unit volume on the particle that was at position \( \vec{x} \) at some reference time. An important case of a body force – a force applied impulsively to one particle at \( \vec{x} = \vec{\xi} \) and \( t = \tau \) in the direction of the \( x_n \)-axis

\[
\begin{align*}
 f_i(\vec{x}, t) &= A\delta(\vec{x} - \vec{\xi})\delta(t - \tau)\delta_{in} \\
 [f_i]^U &= Nm^{-3}, \quad [\delta(\vec{x} - \vec{\xi})]^U = m^{-3} \\
 [A]^U &= Ns, \quad [\delta(t - \tau)]^U = s^{-1}
\end{align*}
\]

1.3 Stress, traction

If forces are applied at a surface \( S \) surrounding some volume of continuum, that volume of continuum is in a condition of stress. This is due to internal contact forces acting mutually between adjacent particles within a continuum. Consider an internal surface \( S \) dividing a continuum into part \( A \) and part \( B \).

\[
\begin{align*}
 \vec{n} &- \text{ unit normal vector to } S \\
 \delta \vec{F} &- \text{ an infinitesimal force acting across an infinitesimal area } \delta S \\
 &- \text{ force due to material } A \text{ acting upon material } B \\
 \vec{T}(\vec{n}) &= \lim_{\delta S \to 0} \frac{\delta \vec{F}}{\delta S} \\
 [\vec{T}]^U &= Nm^{-2}
\end{align*}
\]

\( \vec{T}(\vec{n}) \) – traction vector (stress vector)
- force per unit area exerted by the material in the direction of \( \vec{n} \) across the surface

The part of \( \vec{T} \) – that is normal to the surface – normal stress
- that is parallel to the surface – shear stress

Traction depends on the orientation of the surface element \( \delta S \) across which contract force acts.

Examples:
1.4 Displacement, strain

The state of stress at a point has to be described by a tensor.

1.4 Displacement, strain

Lagrangian description follows a particular particle that is specified by its original position at some reference time. Eulerian description follows a particular spatial position and thus whatever particle that happens to occupy that position. Consider the Lagrangian description.

Displacement $\vec{u} = \vec{u}(\vec{x}, t)$ is the vector distance of a particle at time $t$ from the position $\vec{x}$ of the particle at some reference time $t_0$. $\vec{X} = \vec{x} + \vec{u}$ is the new position.

$$[\vec{u}]^U = m$$

$$\frac{\partial \vec{u}}{\partial t}$$ – particle velocity, $$\frac{\partial^2 \vec{u}}{\partial t^2}$$ – particle acceleration
\( \ddot{u} \) can generally include both the deformation and rigid–body translation and rotation. To analyze the deformation, we compare displacements of two neighboring particles.

\[
\vec{D} = \ddot{u}(\vec{x} + \vec{d}) - \ddot{u}(\vec{x}) \quad (1.3a)
\]

\[
D_i = u_i(x_j + d_j) - u_i(x_j) \quad (1.3b)
\]

\[
u_i(x_j + d_j) = u_i(x_j) + u_{i,j}d_j \quad (1.4a)
\]

\[
 \ddot{u}(\vec{x} + \vec{d}) = \ddot{u}(\vec{x}) + (\ddot{d} \cdot \nabla)\ddot{u}(\vec{x}) \quad (1.4b)
\]

\[
D_i = u_{i,j}d_j \quad (1.5)
\]

\[
u_{i,j} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad (1.6)
\]

\[
e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \quad \text{symmetric tensor} \quad (1.7a)
\]

\[
\Omega_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \quad \text{antisymmetric tensor} \quad (1.8a)
\]

\[
e_{ij} = \begin{bmatrix}
u_{1,1} & \frac{1}{2}(u_{1,2} + u_{2,1}) & \frac{1}{2}(u_{1,3} + u_{3,1}) \\
\frac{1}{2}(u_{2,1} + u_{1,2}) & u_{2,2} & \frac{1}{2}(u_{2,3} + u_{3,2}) \\
\frac{1}{2}(u_{3,1} + u_{1,3}) & \frac{1}{2}(u_{3,2} + u_{2,3}) & u_{3,3}
\end{bmatrix} \quad (1.7b)
\]

\[
\Omega_{ij} = \begin{bmatrix}0 & \frac{1}{2}(u_{1,2} - u_{2,1}) & \frac{1}{2}(u_{1,3} - u_{3,1}) \\
\frac{1}{2}(u_{2,1} - u_{1,2}) & 0 & \frac{1}{2}(u_{2,3} - u_{3,2}) \\
\frac{1}{2}(u_{3,1} - u_{1,3}) & \frac{1}{2}(u_{3,2} - u_{2,3}) & 0
\end{bmatrix} \quad (1.8b)
\]

\[
D_i = e_{ij}d_j + \Omega_{ij}d_j \quad (1.9)
\]

Consider 2D case – a square in the \( xz \)-plane.

Let \( e_{ij} = 0 \). Then \( u_{1,3} = -u_{3,1} \) and \( \Omega_{ij} = \begin{bmatrix}0 & u_{1,3} \\ -u_{1,3} & 0\end{bmatrix} \quad (u_{2,j} = u_{i,2} = 0) \)
1.5 Stress tensor, equation of motion

Let $\Omega_{ij} = 0$ and assume no volume change. Then $u_{1,3} = u_{3,1}$ and $e_{ij} = \begin{bmatrix} 0 & u_{1,3} \\ u_{1,3} & 0 \end{bmatrix}$

Generally,

$\frac{1}{2}(u_{i,j} - u_{j,i})d_j = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{jlm}u_{m,l}d_k = \frac{1}{2}(\text{rot } \vec{u} \times \vec{d})_i$  \hspace{1cm} (1.10)

$\frac{1}{2}$ rot $\vec{u}$ represents a rigid – body rotation if $|u_{i,j}| \ll 1$.

Thus, $e_{ij}$ represents deformation. Therefore, $e_{ij}$ is called the strain tensor. \[ e_{ij} = |U| = |u_{i,j} | \hspace{1cm} (1.11) \]

Displacement is a local measure of an absolute change in position.
Strain is a local measure of relative change in position and displacement field due to deformation.

Example:

Consider a volume $V$ with surface $S$.

\[ \text{time rate of change of momentum of particles} = \text{forces acting on particles} \]
\[ \frac{\partial}{\partial t} \iiint_V \rho \frac{\partial \vec{u}}{\partial t} \, dV = \iiint_V \vec{f} \, dV + \iint_S \vec{T} \, dS \]  
\hspace{1cm} (1.14)

Since \( V \) and \( S \) move with the particles (Lagrangian description), \( \rho dV \) does not change with time and

\[ \frac{\partial}{\partial t} \iiint_V \rho \frac{\partial \vec{u}}{\partial t} \, dV = \iiint_V \rho \frac{\partial^2 \vec{u}}{\partial t^2} \, dV \]  
\hspace{1cm} (1.15)

Consider a particle \( P \) inside the volume \( \Delta V \) for which none of acceleration, body force and traction have singular value. Shrink \( \Delta V \) down onto \( P \) and compare relative magnitudes of the terms in equation (1.14). Both the volume integrals are of order \( \Delta V \) while the surface integral is of order \( (\Delta V)^{\frac{2}{3}} \). This means that the surface integral approaches zero more slowly than the volume integral does. Then (1.14) \( \iint_S dS \) leads to

\[ \lim_{\Delta V \to 0} \frac{\iint_S \vec{T} \, dS}{\iint_S dS} = \lim_{\Delta V \to 0} O(\Delta V^{\frac{1}{3}}) = 0 \]  
\hspace{1cm} (1.16)

Apply equation (1.16) to two cases.

**1st case**

Let \( \Delta V \) be a disc with a negligibly small area of the edge

Equation (1.16) \( \Rightarrow \)

\[ \lim_{\Delta V \to 0} \frac{[\vec{T}(\vec{n}) + \vec{T}(-\vec{n})]}{2S} = 0 \]

\[ \Rightarrow \vec{T}(-\vec{n}) = -\vec{T}(\vec{n}) \]  
\hspace{1cm} (1.17)

**2nd case**

Let \( \Delta V \) be a tetrahedron

Equation (1.16) \( \Rightarrow \)

\[ \lim_{\Delta V \to 0} \frac{\vec{T}(\vec{n}) \cdot ABC + \vec{T}(-\hat{x}_1) \cdot OBC + \vec{T}(-\hat{x}_2) \cdot OCA + \vec{T}(-\hat{x}_3) \cdot OAB}{ABC + OBC + OCA + OAB} = 0 \]  
\hspace{1cm} (1.18)
1.5 Stress tensor, equation of motion

Since \( \vec{n} = (n_1, n_2, n_3); \)
\[ n_1 = \frac{OBC}{ABC}, \quad n_2 = \frac{OCA}{ABC}, \quad n_3 = \frac{OAB}{ABC} \]  \hspace{1cm} (1.19)

and
\[ \vec{T}(-\hat{x}_i) = -\vec{T}(\hat{x}_i); \quad i = 1, 2, 3 \]

we get from (1.18) after dividing it by ABC
\[ \lim_{\Delta V \to 0} \frac{\vec{T}(\vec{n}) - \vec{T}(\hat{x}_1)n_1 - \vec{T}(\hat{x}_2)n_2 - \vec{T}(\hat{x}_3)n_3}{ABC + OBC + OCA + OAB} = 0 \]

and consequently
\[ \vec{T}(\vec{n}) = \vec{T}(\hat{x}_j)n_j \]  \hspace{1cm} (1.20)

Both properties (1.17) and (1.20) are important since they are valid in a dynamic case. (Their validity in a static case is trivial.)

Equation (1.20) can be written as
\[ [T_1(\vec{n}), T_2(\vec{n}), T_3(\vec{n})] = [n_1, n_2, n_3] \begin{bmatrix} T_1(\hat{x}_1) & T_2(\hat{x}_1) & T_3(\hat{x}_1) \\ T_1(\hat{x}_2) & T_2(\hat{x}_2) & T_3(\hat{x}_2) \\ T_1(\hat{x}_3) & T_2(\hat{x}_3) & T_3(\hat{x}_3) \end{bmatrix} \]  \hspace{1cm} (1.21)

Define stress tensor \( \tau_{ji} \)
\[ \tau_{ji} = T_i(\hat{x}_j) \]  \hspace{1cm} (1.22)

\[ [\tau_{ji}]^U = Nm^{-2} \]

Then (1.20) and (1.21) can be rewritten as
\[ T_i(\vec{n}) = \tau_{ji}n_j \quad - \text{Cauchy’s stress formula} \]  \hspace{1cm} (1.23)

and
\[ [T_1(\vec{n}), T_2(\vec{n}), T_3(\vec{n})] = [n_1, n_2, n_3] \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \]  \hspace{1cm} (1.24)

\( \tau_{ji} \) is the \( i \)-th component of the traction exerted by a material with greater \( x_j \) across the plane normal to the \( j \)-th axis on material with lesser \( x_j \).
Stress tensor fully describes a state of stress at a given point.

Now we can apply eq. (1.23) to eq. (1.14). Eq. (1.14) in the index notation is

\[
\iint_V \rho u_{i,tt} \, dV = \iint_V f_i \, dV + \int_S T_i(\vec{n}) \, dS
\]  

(1.25)

Using eq. (1.23) the surface integral becomes

\[
\int_S \tau_{ji} n_j \, dS = \int_S \tau_{ji} \, dS_j
\]  

(1.26)

The surface integral can be transformed into a volume integral using Gauss’s divergence theorem

\[
\iint_S \vec{a} \, dS = \iiint_V \text{div} \, \vec{a} \, dV
\]

\[
\int_S a_j \, dS_j = \iiint_V \frac{\partial a_j}{\partial \xi_j} \, dV(\vec{\xi})
\]

In our problem, the particles constituting S have moved from their original positions \( \vec{x} \) at the reference time to position \( \vec{X} = \vec{x} + \vec{u} \) at time t. Therefore, the spatial differentiation in volume \( V \) is \( \frac{\partial}{\partial \vec{X}_j} \). The application of Gauss’s theorem thus gives

\[
\int_S \tau_{ji} \, dS_j = \iiint_V \frac{\partial \tau_{ji}}{\partial X_j} \, dV
\]  

(1.27)

Eq. (1.25) can be now written as

\[
\iiint_V (\rho u_{i,tt} - f_i - \tau_{ji,j}) \, dV = 0
\]  

(1.28)

where \( \tau_{ji,j} = \partial \tau_{ji}/\partial X_j \)

The integrand in (1.28) must be zero everywhere where it is continuous. Therefore,

\[
\rho u_{i,tt} = \tau_{ji,j} + f_i
\]  

(1.29)

This is the equation of motion for the elastic continuum. Look now at the angular momentum of the particles in a volume \( V \).
1.5 Stress tensor, equation of motion

\[
\begin{align*}
\text{time rate of change of angular} & \quad \text{moment of forces (torque)} \\
\text{momentum about the origin} & \quad \text{acting on the particles} \\
\frac{\partial}{\partial t} \iint \vec{X} \times \rho \vec{u} \, dV & = \iint \vec{X} \times \vec{f} \, dV + \oint \vec{X} \times \vec{T} \, dS \\
(1.30)
\end{align*}
\]

Then eq. (1.30) ⇒

\[
\iint \varepsilon_{ijk} X_j (\rho u_{k,tt} - f_k) \, dV = \int_S \varepsilon_{ijk} X_j T_k \, dS
\]

(1.32)

Eq. (1.29) implies

\[
\iint \varepsilon_{ijk} X_j \frac{\partial \tau_{lk}}{\partial X_l} \, dV = \iint \varepsilon_{ijk} X_j (\rho u_{k,tt} - f_k) \, dV
\]

(1.33)

The right-hand side of eq. (1.33) can be replaced by the right-hand side of eq. (1.32)

\[
\iint \varepsilon_{ijk} X_j \frac{\partial \tau_{lk}}{\partial X_l} \, dV = \int_S \varepsilon_{ijk} X_j T_k \, dS
\]

(1.34)

eq (1.23) ⇒

\[
\int_S \varepsilon_{ijk} X_j \tau_{lk} n_l \, dS
\]

Gauss’s theorem ⇒

\[
\frac{\partial}{\partial X_l} (X_j \tau_{lk}) = \delta_{jl} \tau_{lk} + X_j \frac{\partial \tau_{lk}}{\partial X_l} = \tau_{jk} + X_j \frac{\partial \tau_{lk}}{\partial X_l}
\]

Then eq. (1.34) becomes

\[
\iint \varepsilon_{ijk} X_j \frac{\partial \tau_{lk}}{\partial X_l} \, dV = \iint \left( \varepsilon_{ijk} \tau_{jk} + \varepsilon_{ijk} X_j \frac{\partial \tau_{lk}}{\partial X_l} \right) \, dV
\]

This gives

\[
\iint \varepsilon_{ijk} \tau_{jk} \, dV = 0
\]

(1.35)

Since eq. (1.35) applies to any volume

\[
\varepsilon_{ijk} \tau_{jk} = 0
\]

(1.36)

and consequently

\[
\tau_{jk} = \tau_{kj}
\]

(1.37)
which means that the stress tensor is symmetric. This is a very important property meaning that the stress tensor has only 6 independent components. The state of stress at a given point is thus fully described by 6 independent components of the stress tensor.

We can now rewrite relation for traction (1.23) and equation of motion (1.29) as

\[ T_i = \tau_{ij} n_j \]  
\[ \rho u_{i,tt} = \tau_{ij,j} + f_i \]

Strictly, \( \tau_{ij,j} = \frac{\partial \tau_{ij}}{\partial X_j} \). In the case of seismic wave propagation, displacement, strain, acceleration and stress vary over distances much larger than the amplitude of particle displacement and the other quantities. Therefore, differentiation with respect to \( x_j \) gives a very good approximation of differentiation with respect to \( X_j \). In other words, the difference between derivative evaluated for a particular particle (\( \sim \) Lagrangian description) and derivative evaluated at a fixed position (\( \sim \) Eulerian description) is negligible.

### 1.6 Stress - strain relation. Strain - energy function.

The mechanical behavior of a continuum is defined by the relation between the stress and strain. If forces are applied to the continuum, the stress and strain change together according to the stress–strain relation. Such the relation is called the constitutive relation. A linear elastic continuum is described by Hooke’s law which in Cauchy’s generalized formulation reads

\[ \tau_{ij} = c_{ijkl} e_{kl} \]

Each component of the stress tensor is a linear combination of all components of the strain tensor. \( c_{ijkl} \) is the 4th-order tensor of elastic coefficients and has \( 3^4 = 81 \) components.

\[ \tau_{ij} = \tau_{ji} \Rightarrow c_{ijkl} = c_{jikl} \]  
\[ e_{kl} = e_{lk} \Rightarrow c_{ijkl} = c_{ijlk} \]

The symmetry of the stress and strain tensors reduces the number of different coefficients to \( 6 \times 6 = 36 \). A further reduction of the number of coefficients follows from the first law of thermodynamics which will also give a formula for the strain-energy function.

\[ \text{Rate of mechanical work} + \text{Rate of heating} = \text{Rate of increase of kinetic and internal energies} \]
1.6 Stress - strain relation. Strain - energy function.

Rate of mechanical work
\[ \iiint_V \mathbf{f} \cdot \dot{\mathbf{u}} dV + \iiint_S \mathbf{T} \cdot \dot{\mathbf{n}} dS = \iiint_V f_i \dot{u}_i dV + \iiint_S \tau_{ij} \dot{u}_i n_j dS \]

Gauss’s theorem \( \Rightarrow \)
\[ \iiint_V f_i \dot{u}_i dV + \iiint_S \tau_{ij} \dot{u}_i n_j dV = \iiint_V (\mathbf{f} \cdot \dot{\mathbf{u}}) dV + \iiint_S (\mathbf{T} \cdot \dot{\mathbf{n}}) dS \]
equation of motion (1.39) \( \Rightarrow \)
\[ \iiint_V \mathbf{f} \cdot \dot{\mathbf{u}} dV + \iiint_S \tau_{ij} \dot{u}_i n_j dV = \iiint_V (f_i \dot{u}_i + \tau_{ij} \dot{u}_i) dV \]
\[ = \iiint_V (\rho \dot{u}_i + \tau_{ij} \dot{u}_i) dV \]
\[ = \frac{\partial}{\partial t} \iiint_V \frac{1}{2} \rho \dot{u}_i \dot{u}_i dV + \iiint_V \tau_{ij} \dot{e}_{ij} dV \quad (1.44) \]
since \( \tau_{ij} \dot{u}_{ij} = \tau_{ij} \dot{e}_{ij} \) (antisymmetric part of \( \dot{u}_{ij} \) does not contribute)

Rate of heating
Let \( \dot{\mathbf{h}}(\mathbf{x}, t) \) be the heat flux per unit area and \( Q(\mathbf{x}, t) \) the heat input per unit volume. Then
\[ - \iiint_S \mathbf{h} \cdot \mathbf{n} dS = \frac{\partial}{\partial t} \iiint_V Q dV \]
\[ - \iiint_S h_i n_i dS \]
\[ - \iiint_V h_{i,i} dV = \iiint_V Q dV \quad (1.45) \]
\[ - h_{i,i} = \dot{Q} \quad \text{or} \quad - \nabla \cdot \mathbf{h} = \dot{Q} \quad (1.46) \]

Rate of increase of kinetic energy
\[ \frac{\partial}{\partial t} \iiint_V \frac{1}{2} \rho \dot{u}_i \dot{u}_i dV \quad (1.47) \]

Rate of increase of internal energy
Let \( \mathcal{U} \) be the internal energy per unit volume. Then the rate is
\[ \frac{\partial}{\partial t} \iiint_V \mathcal{U} dV \quad (1.48) \]

Inserting (1.44), (1.45), (1.47) and (1.48) into (1.43) we get
\[ \frac{\partial}{\partial t} \iiint_V \frac{1}{2} \rho \dot{u}_i \dot{u}_i dV + \iiint_V \tau_{ij} \dot{e}_{ij} dV - \iiint_V h_{i,i} dV \]
\[ = \frac{\partial}{\partial t} \iiint_V \frac{1}{2} \rho \dot{u}_i \dot{u}_i dV + \frac{\partial}{\partial t} \iiint_V \mathcal{U} dV \]
This gives
\[ \dot{U} = -h_{i,i} + \tau_{ij} \dot{e}_{ij} \]  
(1.49a)

or (due to (1.46))
\[ \dot{U} = \dot{Q} + \tau_{ij} \dot{e}_{ij} \]  
(1.49b)

For small perturbations of the thermodynamic equilibrium (1.49b) gives
\[ dU = dQ + \tau_{ij} de_{ij} \]  
(1.50)

For reversible processes (1.50) implies
\[ dU = TdS + \tau_{ij} de_{ij} \]  
(1.51)

where \( \tau \) is the absolute temperature and \( S \) entropy per unit volume.

It follows from (1.51) that the entropy and strain–tensor components completely and uniquely determine the internal energy, i.e., they are the state variables.

Define the strain-energy function \( W \) such that
\[ \tau_{ij} = \frac{\partial W}{\partial e_{ij}} \]  
(1.52)

Then (1.51) implies
\[ \tau_{ij} = \left( \frac{\partial U}{\partial e_{ij}} \right)_{S} \]  
(1.53)

If the process of deformation is adiabatic, i.e., if \( \vec{h} = 0 \) and \( \dot{Q} = 0 \), the entropy \( S \) is constant and the internal energy \( U \) can be taken as the strain energy function : \( W = U \).

Since the time constant of thermal diffusion in the Earth is very much longer than the period of seismic waves, the process of deformation due to passage of seismic waves can be considered adiabatic.

Note that the free energy \( F = U - TS \) would be a proper choice for \( W \) in the case of an isothermal process – such as a tectonic process in which the deformation is very slow.

(1.40) and (1.52) :
\[ \frac{\partial W}{\partial e_{ij}} = \tau_{ij} = c_{ijkl} e_{kl} \]  
(1.54)

\[ \frac{\partial^2 W}{\partial e_{kl} \partial e_{ij}} = c_{ijkl} \]

\[ \frac{\partial W}{\partial e_{kl}} = \tau_{kl} = c_{kl ij} e_{ij} \]

\[ \frac{\partial^2 W}{\partial e_{ij} \partial e_{kl}} = c_{kl ij} \]

\[ \Rightarrow c_{ijkl} = c_{kl ij} \]  
(1.55)

Thus, the first law of thermodynamics implies further reduction of the number of independent coefficients – to 21. This is because :

In 6 cases there is \( ij = kl \) and relation (1.55) is identically satisfied.
The remaining 30 \((= 36 - 6)\) coefficients satisfy 15 relations (1.55) which means that only 15 are independent. Thus, \(6 + 15 = 21\). 21 independent elastic coefficients describe an anisotropic continuum in which material parameters at a point depend on direction.

In the simplest, isotropic, continuum, material parameters depend only on position – they are the same in all directions at a given point.

In the isotropic continuum \(c_{ijkl}\) must be isotropic\(^1\):

\[
c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk})
\]  
(1.56)

and \(\lambda\) and \(\mu\) are the only two independent elastic coefficients. Since \(c_{ijkl}\) do not depend on strain, they are also called elastic constants. \(\lambda\) and \(\mu\) are Lamè constants.

Inserting (1.56) into (1.40) gives

\[
\tau_{ij} = \lambda \delta_{ij} e_{kk} + 2 \mu e_{ij}
\]  
(1.57)

which is the Hooke’s law for the isotropic continuum.

Now return to the strain–energy function.

Since all the first derivatives of \(W\) are homogeneous functions (of order one) in the strain–tensor components, and \(W\) can be taken as zero in the natural state\(^2\), \(W\) itself has to be homogeneous (of order two)

\[
W = d_{ijkl} e_{ij} e_{kl}
\]  
(1.58)

\[
\tau_{ij} = \frac{\partial W}{\partial e_{ij}} = d_{ijkl} \left( e_{kl} + e_{ij} \frac{\partial e_{kl}}{\partial e_{ij}} \right)
\]

\[
= d_{ijkl} (e_{kl} + e_{ij} \delta_{ik} \delta_{jl})
\]

\[
= d_{ijkl} (e_{kl} + e_{kl})
\]

\[
= 2d_{ijkl} e_{kl}
\]

(1.40) \(\Rightarrow\)

\[
d_{ijkl} = \frac{1}{2} c_{ijkl}
\]

\[
W = \frac{1}{2} c_{ijkl} e_{ij} e_{kl}
\]

\[
W = \frac{1}{2} \tau_{ij} e_{ij}
\]  
(1.59)

\(^1\) general isotropic tensor : \(c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk});\) due to symmetry of \(c_{ijkl}\) \(\mu = \nu\)

\(^2\) natural state: \(e_{ij} = 0\) and \(W = 0\) \(\Rightarrow\) \(W = d_{ijkl} e_{ij} e_{kl} + \text{constant} = 0\)
Self-gravitation in the Earth causes pressures of up to \( \approx 10^{11} Pa = 10^{11} N m^{-2} \) (the pressure in the center of the Earth reaches \( 4 \cdot 10^{11} Pa \)) and large strains. A finite strain and nonlinear stress–strain relation would be appropriate.

If we want to use theory based on the assumption of small perturbations of a reference state with zero stress and strain (i.e., the assumption used above), we can consider the static equilibrium prior to an earthquake as a reference state. Then we assume zero strain with nonzero initial stress \( \sigma^0_{ij} \), and nonzero strain is then due to incremental stress \( \tau_{ij} \) (the total stress being \( \sigma^0_{ij} + \tau_{ij} \)) and \( \tau_{ij} = c_{ijkl} e_{kl} \).

Unless said otherwise, we will neglect the initial stress \( \sigma^0_{ij} \).

1.7 Uniqueness theorem

The displacement \( \vec{u} = \vec{u}(\vec{x}, t) \) in the volume \( V \) with surface \( S \) at any time \( t > t_0 \) is uniquely determined by

I. \( \vec{u}(\vec{x}, t_0) \) and \( \dot{\vec{u}}(\vec{x}, t_0) \) – initial displacement and particle velocity
II. \( \vec{f}(\vec{x}, t) \) and \( Q(\vec{x}, t) \) – body forces and supplied heat
III. \( \vec{T}(\vec{x}, t) \) over any part \( S_1 \) of \( S \) – traction
IV. \( \vec{u}(\vec{x}, t) \) over \( S_2 \) where \( S_1 + S_2 = S \) – displacement

Proof

Let \( \vec{u}_1(\vec{x}, t) \) and \( \vec{u}_2(\vec{x}, t) \) be any solutions satisfying the same conditions I.–IV. Then, obviously, displacement \( \vec{U}(\vec{x}, t) \equiv \vec{u}_1(\vec{x}, t) - \vec{u}_2(\vec{x}, t) \) has zero initial conditions and is set up by zero body forces, zero heating, zero traction over \( S_1 \) and zero displacement over \( S_2 \).

We have to show that \( \vec{U}(\vec{x}, t) = 0 \) in the volume \( V \) for times \( t > t_0 \).

The rate of mechanical work is obviously zero in \( V \) and on \( S \) for \( t > t_0 \) (\( \vec{f} \equiv 0, \vec{T} \equiv 0 \) on \( S_1 \) and \( \dot{\vec{u}} \equiv 0 \) on \( S_2 \) for \( \vec{U} \)), i.e., according to eq. (1.44)

\[
\frac{\partial}{\partial t} \iiint_V \frac{1}{2} \rho \dot{\vec{U}}_i \dot{U}_i dV + \iiint_V \tau_{ij} \dot{e}_{ij} dV = 0
\]

Integrate the equation from \( t_0 \) to \( t \):

\[
\int_{t_0}^{t} \left[ \frac{\partial}{\partial t} \iiint_V \frac{1}{2} \rho \dot{U}_i \dot{U}_i dV \right] dt = \left[ \iiint_V \frac{1}{2} \rho \dot{U}_i \dot{U}_i dV \right]_{t_0}^{t} = \iiint_V \frac{1}{2} \dot{U}_i(t) \rho \dot{U}_i(t) dV
\]

\[
\int_{t_0}^{t} \left[ \iiint_V \tau_{ij} \dot{e}_{ij} dV \right] dt = \iiint_V \left[ \int_{t_0}^{t} \tau_{ij} \dot{e}_{ij} dt \right] dV
\]
\int_{t_0}^{t} \tau_{ij} \dot{e}_{ij} dt = [\tau_{ij} e_{ij}]_{t_0}^{t} - \int_{t_0}^{t} \dot{\tau}_{ij} e_{ij} dt

= [c_{ijkl} e_{kl} e_{ij}]_{t_0}^{t} - \int_{t_0}^{t} c_{ijkl} \dot{e}_{kl} e_{ij} dt / c_{ijkl} = c_{kl ij}

- \int_{t_0}^{t} c_{ijkl} \dot{e}_{ij} e_{kl} dt

- \int_{t_0}^{t} c_{ijkl} e_{kl} \dot{e}_{ij} dt

- \int_{t_0}^{t} \tau_{ij} \dot{e}_{ij} dt

\int_{t_0}^{t} \tau_{ij} \dot{e}_{ij} dt = [c_{ijkl} e_{kl} e_{ij}]_{t_0}^{t} - \int_{t_0}^{t} \tau_{ij} \dot{e}_{ij} dt

\int_{t_0}^{t} \tau_{ij} \dot{e}_{ij} dt = \frac{1}{2} [c_{ijkl} e_{kl} e_{ij}]_{t_0}^{t}

= \frac{1}{2} [c_{ijkl} U_{k,l} U_{i,j}]_{t_0}^{t}

= \frac{1}{2} c_{ijkl} U_{k,l}(\vec{x},t) U_{i,j}(\vec{x},t)

The integrated equation gives

\iiint_{V} \frac{1}{2} \rho \dot{U_i} \dot{U_i} dV + \iiint_{V} \frac{1}{2} c_{ijkl} U_{k,l} U_{i,j} dV = 0

Since both the kinetic and strain energies are positive, \( \dot{U_i}(\vec{x},t) = 0 \) for \( t \geq t_0 \). Since \( U_i(\vec{x},t_0) = 0 \), \( \vec{U}(\vec{x},t) = 0 \) in \( V \) for \( t > t_0 \).
1.8 Reciprocity theorem

Consider volume $V$ with surface $S$. Let $\mathbf{u} = \mathbf{u}(\mathbf{x}, t)$ be displacement due to body force $\mathbf{f}$, boundary conditions on $S$, and initial conditions at $t = 0$. Let $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ be displacement due to body force $\mathbf{g}$, boundary conditions on $S$, and initial conditions at $t = 0$. Both the boundary and initial conditions are in general different from those for $\mathbf{u}$.

Let $\mathbf{T}(\mathbf{u}, \mathbf{n})$ and $\mathbf{T}(\mathbf{v}, \mathbf{n})$ be tractions due to $\mathbf{u}$ and $\mathbf{v}$, respectively, acting across surface with the normal $\mathbf{n}$.

Then (Betti’s theorem)

\[
\iiint_V \left( f^i - \rho \dddot{u}^i \right) \dddot{v}^i dV + \iint_S \mathbf{T}(\mathbf{u}, \mathbf{n}) \cdot \dddot{v} dS = \iint_V \left( g^i - \rho \dddot{v}^i \right) \dddot{u}^i dV + \iint_S \mathbf{T}(\mathbf{v}, \mathbf{n}) \cdot \dddot{u} dS \tag{1.60}
\]

Proof

\[
\iiint_V \left( f^i - \rho \dddot{u}^i \right) \dddot{v}^i dV + \iint_S T_i(\mathbf{u}, \mathbf{n}) v_i dS - \iint_V \tau_{ij} v_i dV \quad (\text{Eq. of motion (1.39)})
\]

Eq. (1.38) $\Rightarrow$

\[
\iint_S T_i(\mathbf{u}, \mathbf{n}) v_i dS = \iint_S \tau_{ij} n_j v_i dS = \iiint_V (\tau_{ij} v_i)_j dV = \iiint_V \tau_{ij} v_i dV + \iiint_V \tau_{ij} v_i dV = \iiint_V \tau_{ij} v_i dV
\]
Analogously, it can be shown that the right-hand side of eq. (1.60) is equal to

\[ \iiint_V c_{ijkl} u_{ij} v_{kl} dV = \iiint_V c_{kl} c_{ijkl} u_{ij} v_{kl} dV \]

\[ \leftarrow c_{ijkl} = c_{kl} \quad \text{interchanging indices} \]

which is the same as the left-hand side of eq. (1.60)

It is important that

- the theorem does not involve the initial conditions, 
- \( \ddot{u}, \ddot{v}, \dddot{u}, \dddot{v}, T(\ddot{u}, \dddot{n}) \) and \( f \) may relate to time \( t_1 \), while \( \ddot{v}, \dddot{v}, \dddot{T}(\ddot{v}, \dddot{n}) \) and \( \dddot{g} \) may relate to time \( t_2 \neq t_1 \)

Let \( t_1 = t \) and \( t_2 = \tau - t \).

Integrate Betti’s theorem (1.60) from 0 to \( \tau \), integrate first the acceleration terms:

\[ \int_0^\tau \rho \left[ \dddot{u}(t) \cdot \dddot{v}(\tau - t) - \dddot{u}(t) \cdot \dddot{v}(\tau - t) \right] dt \]

\[ = \rho \int_0^\tau \frac{\partial}{\partial t} \left[ \dddot{u}(t) \cdot \dddot{v}(\tau - t) - \dddot{u}(t) \cdot \dddot{v}(\tau - t) \right] dt \]

\[ = \rho \left[ \dddot{u}(\tau) \cdot \dddot{v}(0) - \dddot{u}(0) \cdot \dddot{v}(\tau) + \dddot{u}(\tau) \cdot \dddot{v}(0) + \dddot{u}(0) \cdot \dddot{v}(\tau) \right] \] (1.61)

After the integration, the acceleration terms depend only on the initial \( (t = 0) \) and final \( (t = \tau) \) values.

Let \( \dddot{u} = 0 \) and \( \dddot{v} = 0 \) for \( \tau \leq \tau_0 \). Consequently, also \( \dddot{u} = 0 \) and \( \dddot{v} = 0 \) for \( \tau \leq \tau_0 \).

Then it follows from eq. (1.61) that

\[ \int_{-\infty}^\infty \rho \left[ \dddot{u}(t) \cdot \dddot{v}(\tau - t) - \dddot{u}(t) \cdot \dddot{v}(\tau - t) \right] dt = 0 \] (1.62)

Integrating Betti’s theorem (1.60) from \( -\infty \) to \( \infty \) and applying eq. (1.62) we obtain

\[ \int_{-\infty}^\infty dt \iiint_V \left[ \dddot{u}(x, t) \cdot \dddot{v}(x, \tau - t) - \dddot{u}(x, \tau - t) \cdot \dddot{f}(x, t) \right] dV \]

\[ = \int_{-\infty}^\infty dt \iint_S \left[ \dddot{v}(x, \tau - t) \cdot \dddot{T}(\dddot{u}(x, t), \dddot{n}) - \dddot{u}(x, t) \cdot \dddot{T}(\dddot{v}(x, \tau - t), \dddot{n}) \right] dS \] (1.63)

This is the important reciprocity theorem for displacements \( \dddot{u} \) and \( \dddot{v} \) with a quiescent past.
1.9 Green’s function

Let the unit impulse force in the direction of the \( x_n \) axis be applied at point \( \xi \) and time \( \tau \) (see definition (1.1)):

\[
f_i(\vec{x}, t) = A \delta(\vec{x} - \vec{\xi}) \delta(t - \tau) \delta_{in}
\]

Then the equation of motion is

\[
\rho \ddot{u}_i = (c_{ijkl} u_{k,l})_{,j} + A \delta(\vec{x} - \vec{\xi}) \delta(t - \tau) \delta_{in}
\]

Define Green’s function \( G_{in}(\vec{x}, t; \vec{\xi}, \tau) \):

\[
G_{in}(\vec{x}, t; \vec{\xi}, \tau) = \frac{u_i}{A} ; \quad [G_{in}]^U = \frac{m}{Ns} = \frac{s}{kg}
\]

Green’s function satisfies equation

\[
\rho \ddot{G}_{in} = (c_{ijkl} G_{kn,l})_{,j} + \delta(\vec{x} - \vec{\xi}) \delta(t - \tau) \delta_{in}
\] (1.64)

Let \( A = 1Ns \). Then the value of \( G_{in}(\vec{x}, t; \vec{\xi}, \tau) \) is equal to the value of the \( i \)-th component of the displacement at \( (\vec{x}, t) \) due to the unit impulse force applied at \( (\vec{\xi}, \tau) \) in the direction of axis \( x_n \).

To specify \( G_{in} \) uniquely, we have to specify initial conditions and boundary conditions on \( S \).

Initial conditions:

\[
G_{in}(\vec{x}, t, \vec{\xi}, \tau) = 0 \quad \text{and} \quad \dot{G}_{in}(\vec{x}, t, \vec{\xi}, \tau) = 0 \quad \text{for} \quad t \leq \tau, \vec{x} \neq \vec{\xi}
\]

Boundary conditions on \( S \):

Time independent b.c.

\( \Rightarrow \) The time origin can obviously be arbitrarily shifted. Then eq. (1.64) implies

\[
G_{in}(\vec{x}, t; \vec{\xi}, \tau) = G_{in}(\vec{x}, t - \tau; \vec{\xi}, 0) = G_{in}(\vec{x}, -\tau; \vec{\xi}, -t)
\] (1.65)

Homogeneous boundary conditions (either the displacement or the traction is zero at every point of the surface)

Recall the reciprocity theorem (1.63):

\[
\int_{-\infty}^{\infty} dt \iint_{V} \left[ \vec{u}(\vec{x}, t) \cdot \vec{g}(\vec{x}, \tau - t) - \vec{v}(\vec{x}, t) \cdot \vec{f}(\vec{x}, t) \right] dV
\]

\[
= \int_{-\infty}^{\infty} dt \iint_{S} \left[ \vec{v}(\vec{x}, \tau - t) \cdot \vec{T}(\vec{u}(\vec{x}, t), \vec{n}) - \vec{u}(\vec{x}, t) \cdot \vec{T}(\vec{v}(\vec{x}, \tau - t), \vec{n}) \right] dS
\]
Let \( \mathbf{f} \) and \( \mathbf{g} \) be unit impulse forces

\[
\begin{align*}
f_i(\mathbf{x}, t) & = A\delta(\mathbf{x} - \mathbf{\xi}_1)\delta(t - \tau_1)\delta_{im} \quad (1.66a) \\
g_i(\mathbf{x}, t) & = A\delta(\mathbf{x} - \mathbf{\xi}_2)\delta(t + \tau_2) \quad \text{A} = 1 \text{ Ns} \quad (1.66b)
\end{align*}
\]

Then the displacements \( \mathbf{u} \) due to \( \mathbf{f} \) and \( \mathbf{v} \) due to \( \mathbf{g} \) are

\[
\begin{align*}
u_i(\mathbf{x}, t) & = AG_{im}(\mathbf{x}, t; \mathbf{\xi}_1, \tau_1) \quad (1.67a) \\
v_i(\mathbf{x}, t) & = AG_{in}(\mathbf{x}, t; \mathbf{\xi}_2, -\tau_2) \quad (1.67b)
\end{align*}
\]

Eq. \((1.66b)\) \(\Rightarrow\) \( g_i(\mathbf{x}, \tau - t) = A\delta(\mathbf{x} - \mathbf{\xi}_2)\delta(\tau - t + \tau_2)\delta_{in} \) \( (1.68) \)

Eq. \((1.67b)\) \(\Rightarrow\) \( v_i(\mathbf{x}, \tau - t) = AG_{in}(\mathbf{x}, \tau - t; \mathbf{\xi}_2, -\tau_2) \) \( (1.69) \)

Insert \((1.66a)\), \((1.67a)\), \((1.68)\) and \((1.69)\) into \((1.63)\)

\[
\begin{align*}
\int_{-\infty}^{\infty} dt \iint_V \left[ G_{im}(\mathbf{x}, t; \mathbf{\xi}_1, \tau_1)\delta(\mathbf{x} - \mathbf{\xi}_2)\delta(\tau - t + \tau_2)\delta_{in} \\
- G_{in}(\mathbf{x}, \tau - t; \mathbf{\xi}_2, -\tau_2)\delta(\mathbf{x} - \mathbf{\xi}_1)\delta(t - \tau_1)\delta_{im} \right] dV = 0
\end{align*}
\]

(The integral over \( S \) in \((1.63)\) is zero due to homogeneous boundary conditions.)

\[
\begin{align*}
G_{nm}(\mathbf{\xi}_2, \tau; \mathbf{\xi}_1, \tau_1) & = G_{mn}(\mathbf{\xi}_1, \tau - \tau_1; \mathbf{\xi}_2, -\tau_2) = 0 \\
G_{nm}(\mathbf{\xi}_2, \tau + \tau_2; \mathbf{\xi}_1, \tau_1) & = G_{mn}(\mathbf{\xi}_1, \tau - \tau_1; \mathbf{\xi}_2, -\tau_2) \quad (1.70)
\end{align*}
\]

Let \( \tau_1 = \tau_2 = 0 \). Then eq. \((1.70)\) implies

\[
\begin{align*}
G_{nm}(\mathbf{\xi}_2, \tau; \mathbf{\xi}_1, 0) & = G_{mn}(\mathbf{\xi}_1, \tau; \mathbf{\xi}_2, 0) \quad (1.71)
\end{align*}
\]

Relation \((1.71)\) gives a purely spatial reciprocity of Green’s function.

Example:

\[
\begin{align*}
AG_{12}(\mathbf{\xi}_1, \tau; \mathbf{\xi}_6, 0) & = AG_{12}(\mathbf{\xi}_6, \tau; \mathbf{\xi}_1, 0) \\
AG_{12}(\mathbf{\xi}_2, \tau; \mathbf{\xi}_1, \tau_1) & = AG_{12}(\mathbf{\xi}_1, -\tau_1; \mathbf{\xi}_2, -\tau_2) \quad (1.72)
\end{align*}
\]

Let \( \tau = 0 \). Then eq. \((1.70)\) implies

\[
\begin{align*}
G_{nm}(\mathbf{\xi}_2, \tau_1; \mathbf{\xi}_1, \tau_1) & = G_{mn}(\mathbf{\xi}_1, -\tau_1; \mathbf{\xi}_2, -\tau_2) \quad (1.72)
\end{align*}
\]

Relation \((1.72)\) gives a space–time reciprocity of Green’s function.
1.10 Representation theorem

Find displacement $\vec{u}$ due to body forces $\vec{f}$ in volume $V$ and to boundary conditions on surface $S$ assuming

$$g_i(\vec{x}, t) = A\delta(\vec{x} - \vec{\xi})\delta(t)\delta_{in}$$

and corresponding displacement

$$v_i(\vec{x}, t) = AG_{in}(\vec{x}, t; \vec{\xi}, 0)$$

Insert (1.73) and (1.74) into the reciprocity theorem (1.63)

$$\int_{-\infty}^{\infty} dt \int_{\Omega} \left[ u_i(\vec{x}, t)A\delta(\vec{x} - \vec{\xi})\delta(t)\delta_{in} - AG_{in}(\vec{x}, \tau - t; \vec{\xi}, 0)f_i(\vec{x}, t) \right] dV =$$

$$\int_{-\infty}^{\infty} dt \int_{S} \left[ AG_{in}(\vec{x}, \tau - t; \vec{\xi}, 0)T_i(\vec{u}(\vec{x}, t), \vec{n}) - u_i(\vec{x}, t)T_i \left( AG_{kn}(\vec{x}, \tau - t; \vec{\xi}, 0), \vec{n} \right) \right] dS$$

$$T_i \left( AG_{kn}(\vec{x}, \tau - t; \vec{\xi}, 0), \vec{n} \right) = \tau_{ij}n_j = c_{ijkl}AG_{kn,l}(\vec{x}, \tau - t; \vec{\xi}, 0)n_j$$

Inserting (1.76) into (1.75) we obtain

$$u_i(\vec{\xi}, \tau) = \int_{-\infty}^{\infty} dt \int_{\Omega} f_i(\vec{\xi}, t)G_{in}(\vec{x}, \tau - t; \vec{\xi}, 0)dV$$

$$+ \int_{-\infty}^{\infty} dt \int_{S} \left[ G_{in}(\vec{x}, \tau - t; \vec{\xi}, 0) T_i(\vec{u}(\vec{x}, t), \vec{n}) - u_i(\vec{x}, t)c_{ijkl}n_j \frac{\partial G_{kn}(\vec{x}, \tau - t; \vec{\xi}, 0)}{\partial \xi_l} \right] dS$$

Interchanging formally $\vec{x}$ and $\vec{\xi}$ as well as $t$ and $\tau$ we have

$$\vec{u}_n(\vec{\xi}, t) = \int_{-\infty}^{\infty} d\tau \int_{\Omega} f_i(\vec{\xi}, \tau)G_{in}(\vec{\xi}, t - \tau; \vec{\xi}, 0)dV(\vec{\xi})$$

$$+ \int_{-\infty}^{\infty} d\tau \int_{S} \left[ G_{in}(\vec{\xi}, t - \tau; \vec{x}, 0)T_i(\vec{u}(\vec{\xi}, \tau), \vec{n}) - u_i(\vec{\xi}, \tau)c_{ijkl}(\vec{\xi})n_jG_{kn,l}(\vec{\xi}, t - \tau; \vec{x}, 0) \right] dS(\vec{\xi})$$

Relation (1.77) gives displacement $\vec{u}$ at a point $\vec{x}$ and time $t$ in terms of contributions due to body force $\vec{f}$ in $V$, to traction $\vec{T}$ on $S$ and to the displacement $\vec{u}$ itself on $S$. A disadvantage of
the representation (1.77) is that the involved Green’s function corresponds to the impulse source at $\vec{x}$ and observation point at $\vec{\xi}$.

The reciprocity relations for Green’s function can be used to replace Green’s function in (1.77) by that corresponding to a source at $\vec{\xi}$ and observation point at $\vec{x}$.

Let $S$ be a rigid boundary, i.e., a boundary with zero displacement:

$$G_{\text{rigid}}^{\text{rigid}}(\vec{\xi}, t - \tau; \vec{x}, 0) = 0 \quad \text{for } \vec{\xi} \text{ in } S.$$  

The above condition is a homogeneous condition. Therefore, the spatial reciprocity relation (1.71) can be applied:

$$G_{\text{rigid}}^{\text{rigid}}(\vec{\xi}, t - \tau; \vec{x}, 0) = G_{\text{rigid}}^{\text{rigid}}(\vec{x}, t - \tau; \vec{\xi}, 0)$$

Inserting this into representation relation (1.77) we get

$$u_n(\vec{x}, t) = \int_{-\infty}^{\infty} d\tau \int_{V} \int_{V} f_i(\vec{\xi}, \tau) G_{\text{rigid}}^{\text{rigid}}(\vec{x}, t - \tau; \vec{\xi}, 0) dV(\vec{\xi})$$

$$- \int_{-\infty}^{\infty} d\tau \int_{S} u_i(\vec{\xi}, \tau) c_{ijkl} n_j \frac{\partial G_{\text{rigid}}^{\text{rigid}}}{\partial \xi_l}(\vec{x}, t - \tau; \vec{\xi}, 0) dS(\vec{\xi})$$  \hspace{1cm} (1.78)

Let $S$ be a free surface, i.e., a surface with zero traction:

$$c_{ijkl} n_j \frac{\partial G_{\text{free}}^{\text{free}}}{\partial \xi_l}(\vec{\xi}, t - \tau; \vec{x}, 0) = 0 \quad \text{for } \vec{\xi} \text{ in } S.$$  

This is again a homogeneous condition and relation (1.71) can be applied. It follows from (1.77) that

$$u_n(\vec{x}, t) = \int_{-\infty}^{\infty} d\tau \int_{V} \int_{V} f_i(\vec{\xi}, \tau) G_{\text{free}}^{\text{free}}(\vec{x}, t - \tau; \vec{\xi}, 0) dV(\vec{\xi})$$

$$+ \int_{-\infty}^{\infty} d\tau \int_{S} G_{\text{free}}^{\text{free}}(\vec{x}, t - \tau; \vec{\xi}, 0) T_i \left( \vec{u}(\vec{\xi}, \tau), \vec{n} \right) dS(\vec{\xi})$$  \hspace{1cm} (1.79)